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INFINITESIMALLY CENTRAL EXTENSIONS OF SPIN$_7$ IN CHARACTERISTIC 2

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1. Introduction

1.1. As was mentioned in the introduction of [4], to which we will refer as I, there are infinitesimally central extensions $G^* \to \mathrm{Spin}_7$ in characteristic 2, which are exceptional in that the unipotent radical of G^* is not commutative. (As in I we always assume that the Lie algebra $\mathfrak{g}^*$ of G^* is perfect). Here we will construct a 1-parameter family of such exceptional extensions and thereby fill the gap in the classification results of I. (Compare I: Theorem 11.21, Theorem 13.8.)

1.2. Apart from the methods and results of I we will need a transfer theorem for rational cohomology which was proved in [3]. We reformulate it in 2.7 below.

2. Description of the result

2.1. First we recall some of the setting and the results of I, filling in a few details pertinent to the exceptional extensions considered in this paper. Unexplained notation and terminology is that of I. Let Spin_7 denote the simply connected algebraic group of type B_3 defined and split over $\mathbb{F}_2$. Suppose $\phi: G^* \to \mathrm{Spin}_7$ is a homomorphism such that $d\phi: \mathfrak{g}^* \to \mathfrak{spin}_7$ is a universal central extension of Lie algebras. We are concerned with the classification of the possibilities for (G^*, ϕ), with G^* and ϕ defined over some algebraically closed field K.

Below we will introduce a structure constant $c \in K$ and a finite subgroup Q of the 1-dimensional additive group $\mathbb{G}_a$, such that the pair (c, Q) characterizes (G^*, ϕ) up to isomorphism. We call (c, Q) the *type* of (G^*, ϕ) or $\phi: G^* \to \mathrm{Spin}_7$.

2.2. Given the type (c, Q) of the extension $\phi: G^* \to \mathrm{Spin}_7$, the methods and results of I allow us to give a quite detailed description of G^* (in particular certain subgroups of G^*) and ϕ. For instance, one can derive a presentation for G^* (cf. I, 13.2). In fact, this is how we proved in I (Sections 11, 12 and 13) for the case $c = 0$, that (c, Q)

determines (G^*, ϕ) up to isomorphism. For $c \neq 0$ this uniqueness follows in the same fashion. We leave the details to the reader. (Some of these details are needed for other purposes and can therefore be found below). Here we are interested in the existence problem. We want to show that for each pair (c, Q) there is an extension $\phi: G^* \to \mathrm{Spin}_7$ of type (c, Q). We will see in 2.5 that it suffices to consider the case $Q = 0$. Also, as we proved the existence for type $(0, 0)$ in I (Section 10), we may further restrict ourselves to $c \neq 0$. The case $c \neq 0$, $Q = 0$ turns out to be essentially just one case, i.e. solutions are obtained from a universal solution by specializing c (see 2.5).

2.3. Definition of c. Given $\phi: G^* \to \mathrm{Spin}_7$ we choose a maximal torus T^* in G^* and we choose, for each non-zero weight γ of T^* in $\mathfrak{g}^*$, generators $x_\gamma^*(u)$ as in I (11.6, 11.7). (We reserve the term "root" for the non-zero weights of $\phi(T^*)$, or T^*, in $\mathfrak{spin}_7$.)

Now $c = c_{2,2,-\varepsilon_1-\varepsilon_2,\varepsilon_1+\varepsilon_2+\varepsilon_3}$ is the commutator constant defined by the property

$$(x_\alpha^*(t), x_\gamma^*(u)) = x_{2\alpha+2\gamma}^*(ct^2u^2)x_{\gamma+2\alpha}^*(t^2u)$$

where

$$\alpha = -\varepsilon_1 - \varepsilon_2, \qquad \gamma = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

This constant measures the deviation from "standard" behavior, cf. I (11.25). ("Standard" is the case that $R_u = \ker \phi$ is commutative, cf. I (11.21). In the standard case $c = 0$).

2.4. Definition of Q. For a short root α we define τ^α by

$$(x_\alpha^*(u), x_{-2\alpha}^*(v)) = x_{2\alpha}^*(vu^4)\tau^\alpha(vu^2),$$

cf. I (11.23). It follows as in I (11.27) that the image of τ^α is contained in the center of G^*. One further sees as in I (Section 11) that τ^α is a homomorphism $\mathbb{G}_a \to Z(G^*)$ and that τ^α does not depend on the choice of the root α. (We will often confuse $\mathbb{G}_a$ with the group of its rational points over K, i.e. with the additive group of K). For $\gamma = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ put $\sigma^\gamma(t) = (x_\gamma^*(t), x_{-\gamma}^*(1))$.

Again this defines a homomorphism $\sigma^\gamma: \mathbb{G}_a \to Z(G^*)$ and we have

$$(x_\delta^*(t), x_{-\delta}^*(u)) = \sigma^\gamma(tu) \quad \text{for } \delta = \pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3.$$

Arguing as in I (11.21), cf. I (11.25), we find $\tau^\alpha(ct^2) = \sigma^\gamma(t^2 + t)$ for α, γ, c as above. In particular, $\tau^\alpha(c)$ is the identity of G^*. If $c \neq 0$ we put $x_0^*(t) = \tau^\alpha(ct)\sigma^\gamma(t)$. Then $\tau^\alpha(ct) = x_0^*(t + t^2)$ and $\sigma^\gamma(t) = x_0^*(t^2)$. If $c = 0$ we put $x_0^*(t) = \tau^\alpha(t)$.

(The difference between these two definitions has to do with the fact that the $c = 0$ case differs by an isogeny from what one would get by specializing the construction below, which handles the case $c \neq 0$). As in I (13.1) we now define a finite discrete subgroup Q of $\mathbb{G}_a$ by $Q = \ker x_0^*$.

2.5. The type (c, Q) of $\phi: G^* \to \mathrm{Spin}_7$ depends only on the isomorphism class of G^*. (Recall that ϕ is just the composition of the quotient homomorphism $G^* \to G^*/R_u$ and an isomorphism $G^*/R_u \to \mathrm{Spin}_7$, where R_u is the unipotent radical of G^*. Thus G^* is more interesting than ϕ, cf. I(7.4), I(7.7)).

If we have (G^*, ϕ) of type (c, Q) and if Q' is a finite subgroup of $\mathbb{G}_a$ which contains Q, then $\phi': G^*/x_0^*(Q') \to \mathrm{Spin}_7$ is of type (c, Q'). Here ϕ' is induced by ϕ. So it suffices to prove existence of extensions $\phi: G^* \to \mathrm{Spin}_7$ for type $(c, 0)$.

Theorem 1. *Let G denote the split form of* Spin_7 *over* $S = \mathrm{Spec}(\mathbb{F}_2[X, X^{-1}])$. *There is a homomorphism of group schemes* $\phi_X: G^* \to G$ *over S such that*

(i) *Any specialization homomorphism* $f: \mathbb{F}_2[X, X^{-1}] \to K$ *indices a homomorphism* $\phi_{f(X)}: G^* \times_S \mathrm{Spec}(K) \to \mathrm{Spin}_7$ *of type* $(f(X), 0)$. *In particular,* $\mathrm{d}\phi_{f(X)}$ *is a universal central extension of Lie algebras.*

(ii) *As a scheme,* G^* *is the product over S of G and affine* 15*-space, while* ϕ_X *is the projection onto the factor G in this product.*

In other words, there is a 1-parameter family whose fibers are the desired exceptional extensions. One can extend the family to a family over the projective line, but then one also gets fibers which do not give universal central extensions of $\mathfrak{spin}_7$ (cf. 4.4).

2.6. The proof of Theorem 1 will occupy the remainder of this paper. For simplicity we will not work over S, but simply over K, choosing some non-zero $c \in K$. It will be clear from the construction that it all makes sense over S too, with c replaced by the indeterminate X. The group G^* will be reconstructed "piece by piece". That is, we will prove the existence of certain structures simpler than G^* (like quotients or subgroups) and construct G^* from them.

2.7. We will need a slight generalization of the transfer theorem for rational cohomology which was proved in [3].

Theorem 2 (cf. [3, Theorem 2.1]). *Let G be a connected affine algebraic group, B a Borel subgroup of G, M a rational G-module, all defined over the field k. Then the restriction map* $H^i_k(G, M) \to H^i_k(B, M)$ *is an isomorphism for* $i \geq 0$.

Proof. This was proved in [3] for the case that G is semi-simple. To arrive at the present form of the result we note that, if R is the radical of G,

$$H^0(G, M) \simeq H^0(G/R, H^0(R, M)) \simeq H^0(B/R, H^0(R, M)) \simeq H^0(B, M),$$

where the second isomorphism is an instance of the semi-simple case. Inspecting the proof in [3] we see that the reduction to $i = 0$ is still valid, whence the result.

Alternatively, one may jazz up the above argument for the $i = 0$ case. Namely, by the semi-simple case the spectral sequences

$$E_2^{p,q} = H^p(G/R, H^q(R, M)) \Rightarrow E_\infty^{p+q} = H^{p+q}(G, M)$$

and

$$E_2^{p,q} = H^p(B/R, H^q(R, M)) \Rightarrow E_\infty^{p+q} = H^{p+q}(B, M)$$

are isomorphic. (See [3. Lemma 1.1] for the spectral sequences.)

3. Construction of H and $\tilde{H}$

3.1. Let $\Sigma = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i \neq j \leq 3\}$ denote the root system of Spin_7 and let us order the weights lexicographically. (So $a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 \leq d\varepsilon_1 + e\varepsilon_2 + f\varepsilon_3$ if $a < d$; if $a = d$ and $b < e$; if $a = d$ and $b = e$ and $c \leq f$.)

Let $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha > 0\}$. Let R be the ideal generated by the weight space of ε_1 in the Lie algebra of the adjoint group $\mathrm{S0}_7$. Then R is an irreducible Spin_7-module of dimension 6 with highest weight ε_1. Put

$$P(x \otimes y \otimes z) = x \otimes y \otimes z + z \otimes x \otimes y + y \otimes z \otimes x + y \otimes x \otimes z + z \otimes y \otimes x + x \otimes z \otimes y.$$

(All six permutations occur.) Then P defines a homomorphism of Spin_7-modules $R \otimes R \otimes R \to R \otimes R \otimes R$. Note that $P(X \otimes X \otimes Y) = 0$, because the characteristic is 2. The image $\operatorname{Im} P$ of P is a Spin_7-module with weights $\pm\varepsilon_i$, $\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$. Their multiplicities are 2 and 1 respectively. The irreducible Spin_7-module of highest weight $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ has dimension 8 (e.g. by the proof of I(5.2)).

So $\operatorname{Im} P$ has three composition factors. Computing a few images under the action one sees that they are arranged as follows. The whole module $\operatorname{Im} P$ is indecomposable and there is a unique maximal submodule V of dimension 14. This V is generated by a highest weight vector of weight $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. There is a unique 6 dimensional submodule S of V, isomorphic to R. Of course $\operatorname{Im} P/V$ is also isomorphic to R and V/S is irreducible.

3.2. Let L be the Spin_7-submodule of dimension 7 in the Lie algebra of the adjoint group. Note that L is generated by an element of the Cartan subalgebra that would be called $\frac{1}{2}(H_{\varepsilon_1} + H_{\varepsilon_2} + H_{\varepsilon_3})$ if the characteristic were different from 2, which it isn't. The highest weight of L is ε_1 and L contains R as a submodule, so we have an exact sequence of Spin_7-modules $0 \to V \to \operatorname{Im} P \to L \to K \to 0$, where the field K is viewed as the 1-dimensional Spin_7-module. This resolution of K yields a map $H^0(\mathrm{Spin}_7, K) \to H^2(\mathrm{Spin}_7, V)$. Take a non-trivial 0-cocycle with values in K, or, more specifically, take the one which is defined over $\mathbb{F}_2$. Its image in $H^2(\mathrm{Spin}_7, V)$ is represented by some 2-cocycle of Spin_7 with values in V, defining an extension $1 \to V \to H \to \mathrm{Spin}_7 \to 1$. This defines H.

3.3. Recall that H can be described as a subgroup of the semi-direct product E of $\operatorname{Im} P$ and Spin_7 (see I(9.5)). We fix a maximal torus T of Spin_7 and view T also as a subgroup of E. It can be arranged that $T \subseteq H$, by choosing the cocycles conveniently (within their class), just as in I(10.3). There is a T-equivariant cross section s of $H \to \mathrm{Spin}_7$ (cf. I(10.3, (14)), so we can define elements $y_\alpha(t)$ in H by $y_\alpha(t) = s(x_\alpha(t))$. (Here $x_\alpha(t)$ has the usual meaning, cf. I.) Let X be the set of weights of V, so

$$X = \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3, \pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}.$$

For $\gamma \in X$ let $v_\gamma(1)$ denote the unique weight vector of weight γ in V which is defined over $\mathbb{F}_2$. (The multiplicity of γ is one and $\mathbb{F}_2$ has a unique non-zero element.) The t-multiple of $v_\gamma(1)$ can be viewed as an element of E (or H) and is then written as $v_\gamma(t)$, $(t \in K)$. We now have generators $y_\alpha(t)$ and $v_\gamma(t)$ for H. Because the whole construction can be made quite explicit, as in I(10.3), we can also find relations between these generators. We will list some of these relations below. But let us see first what happens to the generators if s is replaced by another T-equivariant cross section s'. Say $s'(x) = s(x)f(x)$. Then f is a T-equivariant map from Spin_7 into V and one sees from I(11.8) that $f(x_\alpha(t)) = 1$ if $\alpha = \pm\varepsilon_i \pm \varepsilon_j$, but $f(x_\alpha(t)) = v_\alpha(C_\alpha t)$ if $\alpha = \pm\varepsilon_i$. (Here $C_\alpha \in K$.) Inspecting the construction of s in I(10.3) one sees that any combination of constants C_α may actually occur. So if we are not more specific about the choice of s, we must accept some unspecified factors in the relations between the generators. We do accept this but we take s so that it is defined over $\mathbb{F}_2$ (cf. I(8.2) or I(10.3)).

3.4. The group H being an extension of Spin_7 by the module V, it is clear that there are relations of the type

$$y_\alpha(t)v_\gamma(u)y_\alpha(t)^{-1} = \prod_{i \geq 0} v_{\gamma+i\alpha}(C_i t^i u), \quad \text{with } C_i \in \mathbb{F}_2.$$

(The action is defined over $\mathbb{F}_2$.)

Further one has $y_\alpha(t)y_\alpha(u) = y_\alpha(t+u)$, either by direct computation or by the proof of I(11.5). For $h \in T$ one has $hy_\alpha(t)h^{-1} = y_\alpha(\alpha(h)t)$, $hv_\gamma(u)h^{-1} = v_\gamma(\gamma(h)u)$. If $\alpha, \beta \in \Sigma$ with $\alpha + \beta \neq 0$, then

$$(y_\alpha(t), y_\beta(u)) = \Big(\prod_{i,j \geq 0} y_{i\alpha+j\beta}(N_{ij\alpha\beta}t^i u^j)\Big) v_{\alpha\beta}(t, u),$$

where the $N_{ij\alpha\beta}$ are the structure constants of Spin_7 and $v_{\alpha\beta}(t, u)$ is defined as follows. If $\alpha + \beta \notin X$ then $v_{\alpha\beta}(t, u) = 1$. If $\alpha + \beta = \pm\varepsilon_i$, then $v_{\alpha\beta}(t, u) = v_{\alpha+\beta}(C_{\alpha\beta}tu)$ where $C_{\alpha\beta} \in K$ depends on the choice of s, cf. 3.3. If $\alpha + \beta = \pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$, then $v_{\alpha\beta}(t, u) = v_{\alpha+\beta}(tu)$.

3.5. We need a certain central extension $1 \to \mathbb{G}_a \to \tilde{H} \to H \to 1$ of H. To construct it, we restrict to the Borel subgroups and then solve the problem by hand. ("Solvable subgroups can be constructed by hand.") So we now look for an extension

$$1 \to \mathbb{G}_a \xrightarrow{\tilde{v}_0} \tilde{B} \xrightarrow{p} B \to 1,$$

where B is the Borel subgroup of H (i.e. B is generated by T, the $y_\alpha(t)$ with $\alpha \in \Sigma^+$, the $v_\gamma(t)$ with $\gamma \in X$). Before indicating in 3.6 how one proves the existence of this extension, we now discuss it without proofs. The homomorphism $\tilde{v}_0 : \mathbb{G}_a \to \tilde{B}$, or $K \to \tilde{B}$, has its image in the center of $\tilde{B}$. Choose a maximal torus in $p^{-1}(T)$ and identify it with T via p. For $\alpha \in \Sigma^+$, $\gamma \in X$, there are homomorphisms $\tilde{y}_\alpha : K \to \tilde{B}$, $\tilde{v}_\gamma : K \to \tilde{B}$, such that

$$p(\tilde{y}_\alpha(t)) = y_\alpha(t), \qquad p(\tilde{v}_\gamma(t)) = v_\gamma(t),$$
$$h\tilde{y}_\alpha(t)h^{-1} = \tilde{y}_\alpha(\alpha(h)t), \qquad h\tilde{v}_\gamma(t)h^{-1} = \tilde{v}_\gamma(\gamma(h)t),$$

where $h \in T$, $t \in K$. (Compare I, Theorem 8.2.)

Let S be a set of weights. We call it half saturated if $\alpha \in S$, $\beta \in S$, $\beta \geqslant 0$ implies $\alpha + \beta \in S$. For half saturated S we let $\tilde{U}(S)$ be the subgroup of $\tilde{B}$ generated by the $\tilde{y}_\alpha(t)$ with $\alpha \in \Sigma^+ \cap S$ and the $\tilde{v}_\gamma(t)$ with $\gamma \in (X \cup \{0\}) \cap S$.

Put $\tilde{B}(S) = T.\,\tilde{U}(S)$. If S is half saturated and $0 \notin S$, then $\tilde{B}(S)$ is isomorphic with its image under p. So for instance, one has $(\tilde{y}_\alpha(t), \tilde{y}_\beta(u)) = \tilde{v}_{\alpha+\beta}(tu)$ when $\alpha = \varepsilon_2$, $\beta = \varepsilon_1 - \varepsilon_3$. (Choose $S = \{m\alpha + n\beta \mid m \geqslant 0, n \geqslant 0, m + n > 0\}$ and recall that the analogous relation holds in B.)

Similarly $(\tilde{y}_\alpha(t), \tilde{v}_\gamma(u)) = \tilde{v}_{\gamma+2\alpha}(t^2u)$ when $\alpha = \varepsilon_1$, $\gamma = -\varepsilon_1 - \varepsilon_2 + \varepsilon_3$. Etc. Any element of $\tilde{B}$ can be written uniquely in normal form $h\tilde{v}_{\gamma_1}(t_1) \cdots \tilde{v}_{\gamma_{15}}(t_{15})\tilde{y}_{\beta_1}(u_1) \cdots \tilde{y}_{\beta_9}(u_9)$ where $h \in T$, $\gamma_1 < \cdots < \gamma_{15}$, $\gamma_i \in X \cup \{0\}$, $\beta_1 < \cdots < \beta_9$, $\beta_i \in \Sigma^+$. If a factor in the normal form is equal to the identity it will often be deleted. The normal form of an element of $\tilde{U}(S)$, for half saturated S, contains only non-trivial factors whose weights are in S. By means of the co-ordinates t_i, u_i in the normal form one can identify $\tilde{U}(S)$, as a variety, with affine m-space. (m is the number of relevant co-ordinates). Similarly $\tilde{B}$ can be identified, as a variety, with a product of T and affine 24-space. For the group $\tilde{B}$ we get a presentation as follows.

Generators are the $h \in T$, the $\tilde{y}_\alpha(t)$ with $\alpha \in \Sigma^+$ and $t \in K$, the $\tilde{v}_\gamma(t)$ with $\gamma \in X \cup \{0\}$ and $t \in K$.

Relations are: xy = normal form of (xy), where x and y are taken from the generators. For instance

$$\tilde{y}_\alpha(t)\tilde{v}_{-\alpha}(u) = \tilde{v}_{-\alpha}(u)\tilde{v}_0(tu + t^2u^2)\tilde{v}_\alpha(t^2u)\tilde{y}_\alpha(t) \quad \text{for } \alpha = \varepsilon_i;$$

$$\tilde{v}_\alpha(t)\tilde{v}_{-\alpha}(u) = \tilde{v}_{-\alpha}(u)\tilde{v}_\alpha(t) \quad \text{for } \alpha = \varepsilon_i;$$

$$\tilde{v}_\gamma(t)\tilde{v}_{-\gamma}(u) = \tilde{v}_{-\gamma}(u)\tilde{v}_0(tu)\tilde{v}_\gamma(t) \quad \text{for } \gamma = \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3;$$

$$\tilde{y}_\alpha(t)\tilde{y}_\beta(u) = \tilde{v}_{\alpha+\beta}(tu)\tilde{y}_\beta(u)\tilde{y}_\alpha(t) \quad \text{for } \alpha = \varepsilon_2, \beta = \varepsilon_1 - \varepsilon_3.$$

(The last relation follows from what we said before, the other three examples explain how $\tilde{v}_0(t)$ is related to other generators). If S is half saturated one obtains a presentation for $\tilde{B}(S)$ simply by deleting some generators and some relations from the above presentation. (Retain what makes sense in $\tilde{B}(S)$). A presentation for $\tilde{U}(S)$ can be obtained in a similar fashion. If γ is a weight, let $S(\gamma)$ denote the set of weights δ with $\delta \geqslant \gamma$. Then $S(\gamma)$ is half saturated. If $\alpha \in \Sigma^+$, then $\tilde{y}_\alpha(t)$ normalizes $\tilde{U}(S(\gamma))$.

If $\gamma \in X \cup \{0\}$, then $v_\gamma(t)$ normalizes $\tilde{U}(S(\gamma))$. Say $\gamma_1, \ldots, \gamma_{21}$ denote the elements of $\Sigma^+ \cup X \cup \{0\}$ in decreasing order. The subgroups $F_i = \tilde{U}(S(\gamma_i))$ form an increasing filtration of $\tilde{U}$, the unipotent radical of $\tilde{B}$. Let R_i denote the root subgroup of weight γ_i in $\tilde{U}$. (i.e. R_i is generated by the $\tilde{y}_\gamma(t)$, $\tilde{v}_\gamma(t)$ if $\gamma = \gamma_i \in \Sigma^+ \cap X$; it is generated by the $\tilde{y}_\gamma(t)$ if $\gamma = \gamma_i \in \Sigma^+$, $\gamma \notin X$; it is generated by the $\tilde{v}_\gamma(t)$ if $\gamma = \gamma_i \in X \cup \{0\}$, $\gamma \notin \Sigma^+$).

Then F_{i+1} is the semi-direct product (as algebraic group) of its normal subgroup F_i and its subgroup R_{i+1}. $(1 \leqslant i \leqslant 20)$. And $\tilde{B}$ is the semi-direct product of T and the normal subgroup $\tilde{U}$.

3.6. To prove the existence of $\tilde{B}$, and the other facts mentioned in 3.5, one argues inductively, following the filtration $F_1 \subseteq F_2 \cdots \subseteq F_{21} \subseteq \tilde{B}$. From the presentation for $\tilde{B}$ one can obtain explicit formulas for the action of R_{i+1} on F_i, where elements of F_i are written in normal form. So one can tell inductively how to multiply normal forms, by explicit formulas. The question is whether the algebraic structures thus obtained are indeed groups and whether these groups have all the desired properties. Say one has written down the presentation for F_i. Then the first task is to show that F_i exists, i.e. that there is an algebraic group with this presentation and with the desired normal form. (The underlying structure of a variety should of course come from the co-ordinates in the normal form). For $i \leqslant 13$ existence is trivial because F_i can be embedded into B in the obvious way. Say $i \geqslant 13$. To get from existence of F_i to that of F_{i+1} one mainly needs to check that the formulas in the presentation for F_{i+1} do indeed yield an action of R_{i+1} on F_i, so that we can form the semi-direct product. To check that the formulas yield an action it suffices to see that they respect the defining relations for F_i. Now these are all of the form xy = normal form of (xy). Say α, β are the weights of x and y resp. Put

$$S_0 = S(\alpha, \beta, \gamma_{i+1}) = \{r\alpha + s\beta + t\gamma_{i+1} \mid r \geqslant 0, s \geqslant 0, t \geqslant 0, r+s+t > 0\}.$$

If $\tilde{U}(S_0)$ exists, then the action respects the relation xy = normal form of (xy), simply because both the relation and the action on x and y already make sense in $\tilde{U}(S_0)$. (Here "existence" of $\tilde{U}(S_0)$ is to be understood in the same way as existence of F_{i+1}). This way one sees that it suffices to prove existence of the $\tilde{U}(S(\alpha, \beta, \gamma))$ and the $\tilde{B}(S(\alpha, \beta, \beta))$, with

$$S(\alpha, \beta, \gamma) = \{r\alpha + s\beta + t\gamma \mid r \geqslant 0, s \geqslant 0, t \geqslant 0, r+s+t > 0\}.$$

(Such a reduction to three generators is known as the Church Rosser argument, cf. [5, Section 3].) If $S(\alpha, \beta, \gamma)$ does not contain zero, existence is trivial again.

One still has to deal with one of the following four types (or subsystems thereof).

Type 1. $S(\alpha, \gamma, -\gamma)$ where $\alpha \in \Sigma^+$, α is long, $\gamma \in X$, γ is still longer than α and orthogonal to α (e.g. $\alpha = \varepsilon_1 + \varepsilon_3$, $\gamma = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$).

Type 2. $S(\alpha, \beta, -\alpha-\beta)$ where $\alpha = \varepsilon_j$, $\beta \in \Sigma^+$, β is long and orthogonal to α. (So $\beta+\alpha$ and $\beta-\alpha$ are both in X.)

Type 3. $S(\alpha, \beta, -\alpha-\beta)$ where $\alpha = \varepsilon_j$, $b = \varepsilon_k$, $j \neq k$.

Type 4. $S(\alpha, \beta, 0)$ where α, β are long roots making an angle of 120 degrees. For each of these types one can filter as above and construct the desired groups by iterated semi-direct products. The computations are straightforward and will be left to the reader. (The four systems are quite small and can be filtered in various ways, some more convenient than others.)

The second type is the most interesting one because its structure explains the need for the t^2u^2 term in the relation $\tilde{y}_\alpha(t)\tilde{v}_{-\alpha}(u) = \tilde{v}_{-\alpha}(u)\tilde{v}_0(tu+t^2u^2)\tilde{v}_\alpha(t^2u)\tilde{y}_\alpha(t)$, where $\alpha = \varepsilon_j$.

3.7. Now that we have the central extension $1 \to K \to \tilde{B} \to B \to 1$, let us consider the element in $H^2(B, K)$ which represents it. By the Transfer Theorem 2 this element is the restriction of an element of $H^2(H, K)$. So there is an extension $1 \to K \to \tilde{H} \to H \to 1$ of which the above extension is a restriction. The group $\tilde{H}$ is what we were after in this section.

Remark. Originally we proved the existence of the extension $1 \to K \to \tilde{H} \to H \to 1$ by giving an explicit "germ" of a 2-cocycle class, i.e. a function of two variables, defined generically on H and satisfying cocycle conditions (generically) in such a way that Weil's theory of group germs (cf. [1]) applies. The computations needed to check the cocycle conditions were very tedious.

4. The exceptional family

4.1. We will prove Theorem 1 by "mixing" the result of Section 3 with the result of I (Section 10) via a Baer product construction, i.e. by adding 2-cocycles. The constant c will determine the "ratio in the mix".

4.2. Let $\phi: G^* \to \mathrm{Spin}_7$ denote the extension constructed in I (Section 10). (It is of type (0, 0).) The unipotent radical of G^* has, by construction, the structure of a Spin_7-module which is called $\ker \pi$. This $\ker \pi$ is the direct sum of an eight dimensional module and a seven dimensional one. Call the latter N. By I(5.2) one can obtain N from the dual L^* of the module L from 3.2 by applying a Frobenius twist. In the notations of I we have generators $x_\gamma^*(t)$ of G^* and a maximal torus T^* of G^*. Let us identify T^* with T via ϕ so that T has now been identified with a maximal torus in Spin_7, in G^*, in $\tilde{B}$ and hence in $\tilde{H}$.

Choose $c \in K$, $c \neq 0$. Define an endomorphism σ of N by $\sigma(x_\gamma^*(t)) = x_\gamma^*(t)$ for $\gamma = \pm 2\varepsilon_i$, $\sigma(x_0^*(t)) = x_0^*(t(t-c))$. Write $\sigma: N \to \bar{N}$, where the bar symbolizes that σ can be viewed as a quotient map with respect to the discrete subgroup $S =$

$\{x_0^*(0), x_0^*(c)\}$. One has a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, "\sigma"] & G^* \arrow[r] \arrow[d, "p"] & G^*/N \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \bar{N} \arrow[r] & G^*/S \arrow[r] & G^*/N \arrow[r] & 1
\end{tikzcd}$$

where p is the natural projection. Note that the action of G^*/N on $\bar{N}$ is not linear. In fact, there is no G^*/N-equivariant non-trivial endomorphism of $\bar{N}$, as one sees using I(11.9). (Such rigidity does not occur when the action is linear. Then multiplication by a fixed scalar yields an equivariant endomorphism.) As $\bar{N}$ is abelian, the extension $1 \to \bar{N} \to G^*/S \to G^*/N \to 1$ can nevertheless be described by a class of 2-cocycles. (There is a cross section for $G^*/S \to G^*/N$ by the construction of G.) One may obtain a 2-cocycle g in this class by composing σ with the 2-cocycle constructed in I (Section 10). This will make that g enjoys some nice properties, like being T-equivariant and vanishing on T itself.

4.3. Let A be the abelian subgroup of $\tilde{H}$ generated by the $\tilde{v}_\gamma(t)$ with $\gamma = \pm\varepsilon_i$ or $\gamma = 0$. It is easy to see from the uniqueness results of I that $\tilde{H}/A$ is isomorphic with G^*/N [I(13.12)]. We may choose the isomorphism so that it is compatible with the homomorphisms $T \to \tilde{H}/A$, $T \to G^*/N$, $\tilde{H}/A \to \mathrm{Spin}_7$, $G^*/N \to \mathrm{Spin}_7$, induced by homomorphisms introduced before. (Then the isomorphism is unique, cf. I(11.8), (11.17).) So we have an extension $1 \to A \to \tilde{H} \to G^*/N \to 1$. The homomorphism τ from A to $\bar{N}$ which is defined by $\tau(\tilde{v}_\gamma(t)) = x_{2\gamma}^*(ct^2)$ for $\gamma = \pm\varepsilon_i$, $\tau(\tilde{v}_0(t)) = x_0^*(c^2t^2)$, is a G^*/N-equivariant inseparable isogeny. (Note that we use $x_0^*(t)$ to denote both an element of N and an element of $\bar{N}$, which differ as G^*/N-groups. So the same formula would not define a G^*/N-equivariant homomorphism $A \to N$.)

Let us identify T with its image in G^*/N. Applying I(8.2) twice, once to $\tilde{H} \to H$ and once to $H \to G^*/N$ we get a T-equivariant cross section s of $\tilde{H} \to G^*/N$ with $s(T) = T$. It gives rise to a 2-cocycle f for the extension $1 \to A \to \tilde{H} \to G^*/N \to 1$ and via τ we get a 2-cocycle τf of G^*/N with values in $\bar{N}$.

4.4. We now have two 2-cocycles of G^*/N with values in $\bar{N}$, viz. g and τf (see 4.2, 4.3 resp.). Adding them we get a 2-cocycle $g + \tau f$ which describes an extension $1 \to \bar{N} \to G_c^* \xrightarrow[\rho]{} G^*/N \to 1$. We claim that G_c^*, together with the composition of ρ and the projection $G^*/N \to \mathrm{Spin}_7$, yields an extension $G_c^* \to \mathrm{Spin}_7$, of type $(c, 0)$. This will prove Theorem 1.

Remark. If one makes the above construction with $c = 0$ one gets $\tau f = 0$. One may say that $S = \{x_0^*(0), x_0^*(c)\}$ then degenerates to the Lie algebra $\mathfrak{z}$ of the center of G^*, for the following reason. One gets a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, "\sigma"] & G \arrow[r] \arrow[d, "p"] & G^*/N \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \bar{N} \arrow[r] & G^*/\mathfrak{z} \arrow[r] & G^*/N \arrow[r] & 1
\end{tikzcd}$$

where p is the natural quotient map with respect to $\mathfrak{z}$, cf. [2] (17.4). At the other extreme, let us change co-ordinates in such a way that we can put c^{-1} equal to zero. To be precise, use $n_{2\gamma}^*(t) = x_{2\gamma}^*(ct)$ for $\gamma = \pm\varepsilon_i$, $n_0^*(t) = x_0^*(c^2t^2)$. Then $\sigma(x_{2\gamma}^*(t)) = n_{2\gamma}^*(c^{-1}t)$ and $\sigma(x_0^*(t)) = n_0^*(c^{-1}t(c^{-1}t+1))$. So if we put c^{-1} equal to zero then σ vanishes, so g vanishes, and $g + \tau f$ describes the extension $\tilde{H}/\mathfrak{a} \to \mathrm{Spin}_7$, where $\mathfrak{a}$ is the Lie algebra of A. Thus there is a family of extensions, with the projective line as a parameter space, so that the fiber at zero is $G^*/\mathfrak{z}$, the fiber at infinity is $\tilde{H}/\mathfrak{a}$, and in between the fibers are of type G_c^* with $c \neq 0$.

4.5. We still have to see that $\phi_c : G_c^* \to \mathrm{Spin}_7$ is an infinitesimally central extension of type $(c, 0)$. Consider $\rho: G_c^* \to G^*/N$.

The action of G^*/N on $\bar{N}$ is the same as in the extension $1 \to \bar{N} \to G^*/S \to G^*/N \to 1$ of 4.2, so $\mathrm{d}\rho$ is a central extension of Lie algebras. (Note that $\mathrm{d}\sigma$ and $\mathrm{d}p$ are isomorphisms in 4.2. Also compare I(11.2).) Let s be the cross section of ρ for which $s(x)s(y)s(xy)^{-1} = g(x, y) + \tau f(x, y)$. The restriction of s to T is a homomorphism. We identify the torus sT with T. Then s is T-equivariant and we define elements $x_\gamma^*(t)$ in G_c^* by $x_\gamma^*(t) = s(x_\gamma^*(t)/N)$ for $\gamma \in \Sigma$ or $\gamma = \pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$. Recall that we also have elements $x_{2\alpha}^*(t)$ in the subgroup $\bar{N}$ of G_c^* for $\alpha = \pm\varepsilon_i$. To compute an expression like $(x_\alpha^*(t), x_\beta^*(u))$ in G_c^*, where $\alpha = \varepsilon_1 + \varepsilon_2$, $\beta = \varepsilon_1 - \varepsilon_2$, all one has to do is compute its analogue in both $\tilde{H}$ and G^*/S and then multiply the answers in $\bar{N}$. This gives the identity times $x_{\alpha+\beta}^*(tu)$, so $(x_\alpha^*(t), x_\beta^*(u)) = x_{\alpha+\beta}^*(tu)$. From this relation it follows as in I(10.3, (18)) that $\mathrm{d}\rho$ is a universal central extension.

So $\mathrm{d}\phi_c$ is a universal central extension. It is easy to see that the present definitions of the elements $x_\gamma^*(t)$ in G_c^* are compatible with I(Section 11) for $\gamma \neq 0$. Computing in $\tilde{H}$ and G^*/S again we further find $(x_\alpha^*(t), x_\gamma^*(u))x_{\gamma+2\alpha}^*(t^2u) = x_{2\alpha+2\gamma}^*(ct^2u^2)$ for $\alpha = -\varepsilon_1 - \varepsilon_2$, $\gamma = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. So $c = c_{2,2,\alpha,\gamma}$ indeed (cf. 2.3). It is also easy to see that the definition of $x_0^*(t)$ in $\bar{N}$ is not quite compatible with the conventions of 2.4, but that nevertheless $Q = 0$ (in the sense of 2.4). This proves Theorem 1.

Remark. The inseparability of τ is the reason that $g + \tau f$ leads to the same extension of Lie algebras as g does.

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