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On the Schur Multipliers of Steinberg and Chevalley Groups over Commutative Rings

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Introduction

Let Φ be a reduced, irreducible root system [Bo] and A a commutative ring with 1. Define the Steinberg group $\mathrm{St}(\Phi, A)$ by generators and relations as in [Stb] and [S1, §3]. In [S1, §§5 and 6] it was proved that $H_1(\mathrm{St}(\Phi, A), \mathbb{Z}) = H_2(\mathrm{St}(\Phi, A), \mathbb{Z}) = 0$ if the rank of Φ is large enough or if A satisfies certain additional conditions. In particular, $H_1(\mathrm{St}(\Phi, A), \mathbb{Z}) = 0$ whenever $\mathrm{rk}\,\Phi \geqq 3$. In this paper we will compute the exact structure of the Schur multiplier $M(\mathrm{St}(\Phi, A)) = H_2(\mathrm{St}(\Phi, A), \mathbb{Z})$ for such Φ. We need to consider only the five cases $\Phi = A_3, B_3, C_3, D_4, F_4$ since $M(\mathrm{St}(\Phi, A)) = 0$ in all other cases for which $\mathrm{rk}\,\Phi \geqq 3$ [S1, Theorem 5.3]. In two earlier papers we solved this problem $A = \mathbb{Z}$, even when $\Phi = A_2$ [vdK2; S2], and the solution presented here for general A depends on similar calculations. (Unfortunately the calculations for $\mathrm{St}(3, \mathbb{Z}) = \mathrm{St}(A_2, \mathbb{Z})$ don't seem to generalize.)

For each of these five root systems, the problem is reduced to the case of a small ring A: either a finite field or $\mathbb{F}_2[t]/(t^2)$. For finite fields the answers are in the literature [G1]; for $\mathbb{F}_2[t]/(t^2)$ we must find the answer ourselves. This is postponed to §3. (The solution is modeled after [vdK1], so that the computations are tedious.) Along the way we also find the Schur multiplier of $\mathrm{St}(4, A)$ for a (not necessarily commutative) associative ring A with 1. In the last section we display the connection of this work with the Schur multipliers and K_2's of Chevalley groups.

§ 1. Preliminaries

We recall the setting of the computations in [S2], with suitable modifications. Unexplained notation and terminology is that of [S1] or [S2].

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(1.1) *Commutator Formulas. Let G be a group, $a, b, c \in G$, and write ${}^a b = a b a^{-1}$; $[a,b] = {}^a b \cdot b^{-1}$. Then*

(a) $[b,a]^{-1} = [a,b]$,

(b) ${}^b[b^{-1},a] = {}^a[b,a^{-1}] = {}^{ba}[a^{-1},b^{-1}] = {}^{ab}[a^{-1},b^{-1}] = [a,b]$,

(c) $[ab,c] = {}^a[b,c][a,c] = [{}^a b, {}^a c][a,c]$, $\quad [a,bc] = [a,b]{}^b[a,c] = [a,b][{}^b a, {}^b c]$,

(d) $[{}^c a,[b,c]][{}^b c,[a,b]][{}^a b,[c,a]] = 1$ *(Hall's identity).*

(1.2) *Steinberg's Central Trick.* Let $\pi: \tilde{G} \to G$ be a group homomorphism and suppose $\ker(\pi)$ is contained in the center of $\tilde{G}$. Let $a, a', b, b' \in \tilde{G}$ with $\pi(a) = \pi(a')$, $\pi(b) = \pi(b')$. Then $[a,b] = [a',b']$. Typical application of (1.1) and (1.2): If $x, y, z \in \tilde{G}$ with $[\pi(x), \pi(y)] = \pi(z)$, then $[x^2, y] = [{}^x x, {}^x y][x,y] = [x, zy][x,y]$.

(1.3) Now let Φ be a reduced irreducible root system of rank $\geqq 3$ and A a commutative ring with 1. Let $\pi: \tilde{G} \to \mathrm{St}(\Phi, A)$ be the universal central extension of $\mathrm{St}(\Phi, A)$ and choose, for each $\alpha \in \Phi$ and $t \in A$, fixed elements $y_\alpha(t) \in \tilde{G}$ with $\pi(y_\alpha(t)) = x_\alpha(t)$. (The choice of these liftings will be irrelevant so long as they appear inside commutators so that the central trick (1.2) applies.) Recall that $Z_0(\alpha) = \{\beta \in \Phi \mid (a|\beta) = 0 \text{ and } \alpha + \beta \notin \Phi\}$. Define $R \subset \ker(\pi)$ to be the subgroup of $\tilde{G}$ generated by

$$\{[y_\alpha(t), y_\beta(u)] \mid \alpha \in \Phi,\ \beta \in Z_0(\alpha),\ t, u \in A\}.$$

(1.4) **Theorem.** *Assume* $\mathrm{rk}\,\Phi \geqq 3$ *and* $\Phi \neq F_4$. *Then* $R = M(\mathrm{St}(\Phi, A))$. *Moreover it suffices to restrict the generators of R to one fixed α of each length.*

Proof. As in [S2, Theorem 1.5].

(1.5) Let $\varphi: A \to B$ be a surjective ring homomorphism and let Φ be as in (1.4). Then φ induces a surjection $M(\mathrm{St}(\Phi, A)) \to M(\mathrm{St}(\Phi, B))$ sending the generator $[y_\alpha(t), y_\beta(u)]$ to $[y_\alpha(\varphi(t)), y_\beta(\varphi(u))]$. Thus the order of $M(\mathrm{St}(\Phi, B))$ is a lower bound for the order of $M(\mathrm{St}(\Phi, A))$.

(1.6) Suppose $\Phi = F_4$ with simple roots $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$, $\alpha_3 = \varepsilon_4$, $\alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$ [Bo, p. 272]. Then $\{\alpha_1, \alpha_2, \alpha_3\}$ span a root system of type B_3. The following facts are implicit in [S1, §§5, 6]. The inclusion of this B_3-system into F_4 induces a homomorphism $\mathrm{St}(B_3, A) \to \mathrm{St}(F_4, A)$ which factors (uniquely) through the universal central extension $\pi: \tilde{G} \to \mathrm{St}(F_4, A)$. (This is a consequence of the fact that $M(\mathrm{St}(B_3, A))$ has trivial image in $\tilde{G}$ (cf. [S1, Lemma 6.2]). One may also check this using Theorem 2.8 below.) Let $y_\alpha^B(t)$ denote the image in $\tilde{G}$ of the generator $x_\alpha(t)$ of $\mathrm{St}(B_3, A)$. Note that $\pi(y_\alpha^B(t)) = x_\alpha(t) \in \mathrm{St}(F_4, A)$.

Similarly $\{\alpha_2, \alpha_3, \alpha_4\}$ span a root system of type C_3 and there is an induced homomorphism $\mathrm{St}(C_3, A) \to \tilde{G}$. The image in $\tilde{G}$ of the generator $x_\alpha(t)$ of $\mathrm{St}(C_3, A)$ is denoted $y_\alpha^C(t)$. As before, $\pi(y_\alpha^C(t)) = x_\alpha(t) \in \mathrm{St}(F_4, A)$.

Let $R' \subset \ker(\pi)$ be the subgroup of $\tilde{G}$ generated by all $y_\alpha^B(t)\, y_\alpha^C(t)^{-1}$, $t \in A$, where α is in the intersection of the B_3 and C_3 subsystems described above (so $\alpha = \pm\varepsilon_3, \pm\varepsilon_4, \pm\varepsilon_3 \pm \varepsilon_4$).

(1.7) **Theorem.** $R' = M(\mathrm{St}(F_4, A))$. *It suffices to restrict the generators of R' to one fixed α of each length. A surjective ring homomorphism $\varphi: A \to B$ induces a* *surjection* $M(\mathrm{St}(F_4, A)) \to M(\mathrm{St}(F_4, B))$ *sending* $y_\alpha^B(t)\, y_\alpha^C(t)^{-1}$ *to* $y_\alpha^B(\varphi(t))\, y_\alpha^C(\varphi(t))^{-1}$; *thus the order of* $M(\mathrm{St}(F_4, B))$ *is a lower bound for the order of* $M(\mathrm{St}(F_4, A))$.

Proof. As in [S2, Theorem 1.5].

§2. Calculation of the Multipliers

In (2.6), (2.8) and (2.10) we will assume the results of §3. The remainder of this section is independent of §3 and may, therefore, be used in §3. We will not elaborate much on detailed computations when they are analogous to those in [S2].

We will label our root systems as in [Bo, pp. 250–275]. In order to keep our subscripts manageable, we adopt the following notation. The subscript ij stands for the root $\varepsilon_i + \varepsilon_j$. If $\Phi = B_3$ or F_4, the single subscript i denotes the root ε_i; if $\Phi = C_3$, it stands for the root $2\varepsilon_i$. A primed subscript denotes a minus sign: thus $i'j$ stands for $-\varepsilon_i + \varepsilon_j$ and i' stands for $-\varepsilon_i$, except if $\Phi = C_3$ when it stands for $-2\varepsilon_i$. (This notation was inspired by Griess' thesis; cf. [G 2].)

(2.1) We will often use Hall's identity (1.1d) to find a relation in $\tilde{G}$. Relations (1.1)(a)–(c) and the central trick (1.2) can then be used to simplify the relation. The central trick usually allows us to eliminate any repetition of a root within a commutator, e.g.

$$[y_\alpha(t)\, y_\beta(u)\, y_\alpha(t)^{-1}, y_\gamma(v)] = [y_{\alpha+\beta}(t u)\, y_\beta(u), y_\gamma(v)]$$

when $[x_\alpha(t), x_\beta(u)] = x_{\alpha+\beta}(t u)$.

As a rule we consider the right hand sides of (1.1) (a)–(c) to be better than their other members. An exception occurs when $\pi([b, c]) = 1$, so that $[b, c]$ is central. Then, naturally, we prefer $[b, c]\,[a, c]$ over ${}^a[b, c]\,[a, c]$.

Our procedure for simplifying relations is to apply alternately the central trick and the commutator relations. The following fact [S1, Proposition 5.12] allows us to delete many commutators: If $(\alpha|\beta) > 0$, $\alpha \neq \beta$, $\alpha + \beta \notin \Phi$, and rk $\Phi \geqq 3$, then $[y_\alpha(t), y_\beta(u)] = 1$.

(2.2) *The Case* $G = \mathrm{St}(C_3, A)$. We set $P(r, s) = [y_{12'}(r), y_3(s)]$, $Q(r, s) = [y_3(r), y_2(s)]$, for $r, s \in A$. Then the elements $P(r, s)$, $Q(r, s)$ generate $R = M(G)$ (cf. [S2, (2.1)]). The commutator relations and the central trick imply $P(r, s+t) = P(r, s)\, P(r, t)$; $P(r+s, t) = P(r, t)\, P(s, t)$, and similarly for Q. Applying (1.1d) to $a = y_3(r)$, $b = y_{12}(s)$, $c = y_{1'2}(1)$ and simplifying, we obtain $[y_3(r), y_2(\pm 2s)] = 1$, which implies $Q(r, s)^2 = 1$. It follows as in [S2, (2.1)] that $[y_i(r), y_j(s)] = Q(r, s)$ for $i \neq j$. In particular, $Q(r, s) = Q(s, r)$.

Next apply (1.1d) to $a = y_{12}(t)$, $b = y_{2'}(r)$, $c = y_3(s)$. Simplifying, we learn that $P(rt, s) = Q(s, rt^2)$; hence $P(rt, s) = Q(rt^2, s)$. Setting $t = 1$, we have $P(r, s) = Q(r, s)$. Thus $P(r(t^2 - t), s) = 0$.

Finally apply (1.1d) to $a = y_3(1)$, $b = y_{13'}(t)$, $c = y_{2'3}(r)$, to obtain $P(r, t) = P(1, rt)$. Therefore the map sending $a \in A$ to $P(1, a)$ is a homomorphism from the additive group of A onto the Schur multiplier $R = M(G)$. Call this homomorphism φ. Then $\ker(\varphi)$ contains the ideal I generated by $\{t^2 - t \,|\, t \in A\}$, which is the intersection of all maximal ideals $\mathfrak{m}$ of A with $A/\mathfrak{m} \approx \mathbb{F}_2$ [S1, (4.2) (c)]. Set $\bar{A}=A/I$. Thus if $0 \neq e \in \bar{A}$, there is a homomorphism $f\colon \bar{A} \to \mathbb{F}_2$ with $f(e)=1$. This yields a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \bar{A} \arrow[r, "f"] \arrow[d] & \mathbb{F}_2 \arrow[d] \\ M(\mathrm{St}(C_3, A)) \arrow[r] & M(\mathrm{St}(C_3, \bar{A})) \arrow[r] & M(\mathrm{St}(C_3, \mathbb{F}_2)) \end{tikzcd}$$

in which the vertical maps are onto. Since $M(\mathrm{St}(C_3, \mathbb{F}_2)) \neq 0$ [G, p. 70], the right-hand vertical map is an isomorphism, and the image of e in $M(\mathrm{St}(C_3, \bar{A}))$ must be non-zero.

We conclude that $\ker(\varphi)=I$.

Theorem. *There is a homomorphism from the additive group of A onto $M(\mathrm{St}(C_3, A))$ sending $a \in A$ to $[y_{\varepsilon_1-\varepsilon_2}(1), y_{2\varepsilon_3}(a)]$. Its kernel is the ideal generated by $\{t^2-t \mid t \in A\}$.*

Remark. Thus we retrieve the result of [S1, Theorem 5.3]: $M(\mathrm{St}(C_3, A))=0$ if A has no residue field isomorphic to $\mathbb{F}_2$.

(2.3) *The Case* $G=\mathrm{St}(F_4, A)$. Let $P(r)=y_3^B(r)\, y_3^C(r)^{-1}$, $Q(r)=y_{34}^B(r)\, y_{34}^C(r)^{-1}$ for $r \in A$. Then $R'=M(G)$ is generated by $P(r)$, $Q(r)$ (cf. (1.6)). The central trick implies

$$[y_4^B(u), y_{34'}^B(v)]=[y_4^C(u), y_{34'}^C(v)].$$

The left-hand side equals $y_{34}^B(\eta u^2 v)\, y_3^B(\omega u v)$; the right-hand side equals $y_{34}^C(\eta u^2 v) \cdot y_3^C(\omega u v)$, for $\eta = \pm 1$, $\omega = \pm 1$. Hence $P(\omega u v)=Q(\eta u^2 v)^{-1}$ using the centrality of $P(\omega u v)$, which implies $P(\omega v)=Q(\eta v)^{-1}=P(-\omega v)$. It is easy to check that $P(r+s)=P(r)\,P(s)$, so this implies $P(\omega v)^2=1$, $P(\omega v)=P(v)=Q(\eta v)^{-1}$, $P=Q$, $P((u-u^2)v)=1$. Arguing as for C_3 we obtain the following generalization of [S1, Theorem 5.3].

Theorem. *There is a homomorphism from the additive group of A onto $M(\mathrm{St}(F_4, A))$ sending $a \in A$ to $y_{\varepsilon_3}^B(a)\, y_{\varepsilon_3}^C(a)^{-1}$. Its kernel is the ideal generated by $\{t^2-t \mid t \in A\}$.*

(2.4) *The Case* $G=\mathrm{St}(4, A)$. Taking $\Phi=A_3$ yields the group $\mathrm{St}(4, A)$ of algebraic K-theory [M, p. 39]. We prefer the notation $\mathrm{St}(4, A)$ here since the group in question is defined even if A is not commutative. So let A be an associative ring with 1. Theorem (1.4) ($R=M(G)$) still holds with the same proof as for commutative rings.

We use the standard notations of algebraic K-theory in place of the notation introduced at the beginning of this section. (Thus we write x_{ij} for what was previously called $x_{ij'}$; they denote $x_{\varepsilon_i-\varepsilon_j}$.) Let $y_{ij}(r)$ be fixed liftings to $\tilde{G}$ of the generators $x_{ij}(r)$ of G and set $P(r,s)=[y_{12}(r), y_{34}(s)]$. Then the $P(r,s)$, $r, s \in A$, generate $R=M(G)$. As usual, $P(r+s,t)=P(r,t)\,P(s,t)$ and $P(r,s+t)=P(r,s)\,P(r,t)$. Also if i, j, k, l are distinct, standard computations show $[y_{ij}(r), y_{kl}(s)]=P(r,s)^{\pm 1}$.

Now recall that $[y_{ij}(r), y_{ik}(s)]=[y_{ji}(r), y_{ki}(s)]=1$ (this is the situation referred to at the end of (2.1)). Applying (1.1d) with $a=y_{12}(p)$, $b=y_{32}(q)$, $c=y_{24}(r)$, we find that $[y_{12}(p), y_{34}(q\,r)]=[y_{32}(q), y_{14}(-p\,r)]^{-1}$. Both sides are central, so let us conjugate the right-hand side by $y_{13}(1)\,y_{31}(-1)\,y_{13}(1)$ while leaving the left-hand side alone. This yields $[y_{12}(p), y_{34}(q\,r)]=[y_{12}(q), y_{34}(p\,r)]^{-1}$, that is, $P(p, q\,r)=P(q, p\,r)^{-1}$. If we set $F(r)=P(1, r)^{-1}$ and make the proper substitutions, we learn $P(p, r)=F(p\,r)$, $F(p\,q\,r)=F(q\,p\,r)^{-1}$, $F(r+s)=F(r)\,F(s)$, $F(r)^2=1$, $F(p\,q)=F(q\,p)$, $F(p\,q\,r\,s)=F(q\,r\,s\,p)=F(r\,q\,s\,p)=F(p\,r\,q\,s)$. Thus F annihilates the 2-sided ideal J of A generated by all $q\,r-r\,q$ and 2.

Next apply (1.1d) with $a=y_{12}(p)$, $b=y_{24}(q)$, $c=y_{43}(r)$ to obtain

$$[y_{12}(p), y_{23}(q\,r)]\,F(q\,r\,p\,q)\,[y_{43}(r), y_{14}(p\,q)]=1.$$

Substituting 1 for q, $q\,r$ for r and comparing with the original identity yields

$$(*)\quad F(q\,r\,p\,q-q\,r\,p)=[y_{43}(q\,r), y_{14}(p)]\,[y_{43}(r), y_{14}(p\,q)]^{-1}.$$

Substitute $p\,q$ for p, r for q, 1 for r and multiply with $(*)$ to find that

$$F(q\,r\,p\,q-q\,r\,p+r\,p\,q\,r-r\,p\,q)=[y_{43}(q\,r), y_{14}(p)]\,[y_{43}(1), y_{14}(p\,q\,r)]^{-1}.$$

But this also equals $F(q\,r\,p\,q\,r-q\,r\,p)$, as one sees by substituting 1 for r and $q\,r$ for q in $(*)$. Thus we learn that $F(q\,r\,p\,q-q\,r\,p+r\,p\,q\,r-r\,p\,q-q\,r\,p\,q\,r+q\,r\,p)=1$, or, since F annihilates J, $F(p(q^2+q)(r^2+r))=1$. Hence F annihilates the two-sided ideal I of A generated by 2, $q\,r-r\,q$ and $(q^2+q)(r^2+r)$, $q, r\in A$. Let $\bar{A}=A/I$. Then $\bar{A}$ is commutative and $x^4-x^2=(x^2+x)^2=0$ for all x. Thus every square in $\bar{A}$ is an idempotent (possibly 0).

(2.5) Let $\mathbb{F}_2[\varepsilon]=\mathbb{F}_2[t]/(t^2)$, $\varepsilon=t/(t^2)$, be the ring of dual numbers over $\mathbb{F}_2$.

Lemma. *Let $x\in\bar{A}$. Then $x\neq 0$ if and only if there is a homomorphism $\varphi\colon \bar{A}\to\mathbb{F}_2[\varepsilon]$ with $\varphi(x)\neq 0$.*

Proof. Let $x\in\bar{A}$, $x\neq 0$, and let $\mathfrak{p}$ be a prime ideal of $\bar{A}$ containing the annihilator of x, so that the image of x in $\bar{A}_{\mathfrak{p}}$ is non-zero. If $y\in\bar{A}-\mathfrak{p}$, the image of y^2 in $\bar{A}_{\mathfrak{p}}$ is an invertible idempotent in a local ring, hence equals 1. This proves that the localization map $\bar{A}\to\bar{A}_{\mathfrak{p}}$ is surjective (in particular, the ideal of $\bar{A}_{\mathfrak{p}}$ generated by 2 and all $(u^2+u)(v^2+v)$ vanishes) and we may assume that $\bar{A}$ is local with maximal ideal $\mathfrak{m}$. If $x\notin\mathfrak{m}$, its image in the residue field is non-zero and that residue field is $\mathbb{F}_2$ which is a subring of $\mathbb{F}_2[\varepsilon]$. If $x\in\mathfrak{m}$, we argue as follows. Each $u\in\mathfrak{m}$ has square 0 since u^2 is idempotent. Hence $u=u^2+u$ and $\mathfrak{m}^2=(0)$. Since $\bar{A}$ is additively generated by 1 and $\mathfrak{m}$, each additive subgroup of $\mathfrak{m}$ is actually an ideal. And since $\bar{A}$ is an $\mathbb{F}_2$-vector space, there is an ideal $\mathfrak{n}$ which is a complementary vector subspace to the subring generated by 1 and x. Clearly $\bar{A}/\mathfrak{n}$ is isomorphic to $\mathbb{F}_2[\varepsilon]$.

(2.6) We will show in § 3 that if $A=\mathbb{F}_2[\varepsilon]$, the homomorphism $F\colon A\to M(G)$ is injective. Therefore we conclude as in (2.2):

Theorem. *Let A be an associative ring with 1. There is a homomorphism from the additive group of A onto $M(\mathrm{St}(4, A))$ sending $a\in A$ to $[y_{12}(1), y_{34}(a)]$. Its kernel is the two-sided ideal generated by 2, all $q\,r-r\,q$ and all $(q^2+q)(r^2+r)$, $q, r\in A$. In particular, $M(\mathrm{St}(4, A))=1$ if and only if A admits no non-trivial homomorphism to $\mathbb{F}_2$.*

(2.7) *The Case* $G=\mathrm{St}(B_3, A)$. We assume once again that A is commutative and resume our usual notation. The long roots in B_3 form a subsystem of type A_3, which allows us to use the results of (2.5) as follows. Set $F(r)=[y_{12}(1), y_{12'}(r)]$. Then $F(r)F(s)=F(r+s)$; F annihilates the ideal generated by 2 and all (q^2+q) (r^2+r); and $[y_\alpha(r), y_\beta(s)]=F(rs)$ when α and β are long and $(\alpha\,|\,\beta)=0$. Now set $Q(r,s)=[y_1(r), y_{23}(s)]$. One checks as usual that $Q(r+s,t)=Q(r,t)\,Q(s,t)$; $Q(r,s+t)=Q(r,s)\,Q(r,t)$ and that $[y_{\pm\varepsilon_i}(r), y_{\pm\varepsilon_j\pm\varepsilon_k}(s)]=Q(r,s)^{\pm 1}$ if i, j, k are distinct.

Applying (1.1d) to $a=y_1(t)$, $b=y_{1'2}(v)$, $c=y_{13}(1)$ yields $Q(t,v)=Q(\eta, tv)$ for some $\eta=\pm 1$. Substituting 1 for t and tv for v, we deduce that $Q(t,v)=Q(1,tv)$. Set $H(v)=Q(1,v)$. Apply (1.1d) with $a=y_1(t)$, $b=y_{1'3}(1)$, $c=y_{23'}(v)$ and simplify (see (2.1)) to obtain

$$\begin{aligned}&[y_{23'}(v), y_{13}(\pm t^2)]\,[y_1(t), y_{1'2}(\pm v)]\,H(\pm t^3 v)\,H(\pm tv)\,F(t^2v^2)\\ &\quad=[y_{13}(\pm t^2), y_{1'2}(\pm v)]\,[y_3(\pm t), y_{23'}(v)].\end{aligned}$$

If we apply π, the image in G of the left-hand side is $x_{12}(*)\,x_2(\pm tv)$ while the image of the right-hand side is $x_{23}(*')\,x_2(\pm tv)$. Since these must be the same, we conclude that $*=*'=0$ and their common image under π is $x_2(\pm tv)$.

Call the left-hand side of this expression y and let $w=y_{13'}(1)\,y_{1'3}(-1)\,y_{13'}(1)$. Then conjugating y by w is the same as multiplying y on the left by $[w,y]=[w, x_2(\pm tv)]=H(\pm tv)\,H(\pm tv)\,H(\pm tv)$ (by the central trick). Let us conjugate y in this way and also conjugate the right-hand side by w in the ordinary way, i.e. factor by factor. This yields

$$\begin{aligned}(**)\quad &H(\pm tv\pm tv\pm tv\pm tv\pm t^3v)\,F(t^2v^2)[y_{23'}(v), y_{13}(s_1t^2)]\,[y_1(t), y_{1'2}(s_2v)]\\ &\quad=[y_{13}(s_3t^2), y_{23'}(s_4v)]\,[y_1(s_5t), y_{1'2}(s_6v)]\end{aligned}$$

for some set of signs $s_i=\pm 1$. We may assume $s_5=1$ (otherwise use w^{-1} instead of w). It is easy to see that

$$[y_{13}(s_3t^2), y_{23'}(s_4v)]=[y_{23'}(v), y_{13}(s_1t^2)]^{-s_1s_3s_4}.$$

Again projecting to G, we see that $s_2=s_6$ and $s_1s_3s_4=-1$. Hence the commutators in $(**)$ cancel so that $H(\pm tv\pm tv\pm tv\pm tv\pm t^3v)\,F(t^2v^2)=1$. (One can also do the computations with a set of consistent structure constants. Then one sees directly that these signs behave in the way described above.) Thus there is an $N\in\{-4,-2,0,2,4\}$ such that $H(Ntv-t^3v)=F(t^2v^2)$, and this N is independent of the ring A. For $A=\mathbb{Z}$, we know [S2, Theorem 2.6] that $M(G)$ has order 6 and is generated by $H(1)$. Putting $t=1$, $v=2$, we obtain $H(2N-2)=F(4)=1$; hence $N=4$ or $N=-2$. If $N=-2$, put $t=-1$, $v=2u$ to obtain $H(6u)=F(4u^2)=1$. Thus $H(4tv-t^3v)=H(-2tv-t^3v)=F(t^2v^2)$ and we may as well assume that $N=4$. A similar calculation then shows that we still have $H(6u)=1$.

Now set $L(a)=H(2a)\,F(a)$, $a\in A$. Then $L(3a)=F(a)$, $L(3v^2)=F(v^2)=H(3v)$. So L is a homomorphism from the additive group of A onto $M(G)$. The kernel of L contains the ideal I generated by 6, $3(q^2+q)\,(r^2+r)$ and $8t-2t^3$ (setting $v=2u$). The ring $\bar{A}=A/I$ is the product of $\bar{A}/2\bar{A}$ and $\bar{A}/3\bar{A}$. The latter has trivial nilradical. If $A=\mathbb{F}_3$ or $\mathbb{F}_2[\varepsilon]$, the map L is injective (see [G1, p. 70] and § 3).

Hence we conclude:

(2.8) **Theorem.** *There is a homomorphism from the additive group of A onto $M(\mathrm{St}(B_3, A))$ sending $a \in A$ to $[y_{\varepsilon_1}(2), y_{\varepsilon_2+\varepsilon_3}(a)][y_{\varepsilon_1+\varepsilon_2}(1), y_{\varepsilon_1-\varepsilon_2}(a)]$. Its kernel is the ideal generated by 6, $3(q^2+q)(r^2+r)$ and $2q-2q^3 (q, r \in A)$. In particular, $M(\mathrm{St}(B_3, A))=1$ if and only if A admits no non-trivial homomorphisms to $\mathbb{F}_3$ and $\mathbb{F}_2$.*

(2.9) *The Case* $G=\mathrm{St}(D_4, A)$. For $r \in A$, set

$$K(r)=[y_{12'}(1), y_{12}(r)],\ H(r)=[y_{12'}(1), y_{34}(r)],$$

$F(r)=[y_{12'}(1), y_{34'}(r)]$. Using the various subsystems of D_4 of type A_3 and the results of (2.5), we see that F, H and K annihilate the ideal generated by 2 and all $(q^2+q)(r^2+r)$. Moreover, according to (1.4), $\{F(r), H(r), K(r) \mid r \in A\}$ generates $M(G)$. For $\alpha \in D_4$, set $s_\alpha=y_\alpha(1)\, y_{-\alpha}(-1)\, y_\alpha(1)$. Conjugate $y=[y_{13'}(q), y_{32'}(1)]$ in 2 ways by $w=s_{\varepsilon_1+\varepsilon_2} s_{\varepsilon_3+\varepsilon_4} s_{\varepsilon_3-\varepsilon_4}$. (The usual way is factor by factor; the unusual way is left multiplication by $[w, y]=[w, y_{12'}(\pm q)]$. Cf. (2.7).) We obtain

$$\begin{aligned} K(q)\, H(q)\, F(q)\, y &= [y_{32'}(\pm q), y_{13'}(\pm 1)] \\ &= [y_{13'}(1), y_{32'}(q)]^{\pm 1} \end{aligned}$$

and the exponent of the last term must be $+1$ because the images of the 2 sides under π must agree in G. Since $\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_3-\varepsilon_4$ span a subsystem of type A_3, we may apply relation $(*)$ of (2.4). Performing the cyclic permutation (432) on the indices in that relation and substituting $p=r=1$, we see that $[y_{13'}(1), y_{32'}(q)] = y\, F(q^2-q)$. We conclude that $F(q^2)=H(q)\, K(q)$, so that $1=F((r^2+r)^2)= H(r^2+r)\, K(r^2+r)$, or $H(r^2+r)=K(r^2+r)$. Similarly, $F(r^2+r)=K(r^2+r)$, and it follows that $M(G)$ is generated by all $K(q)$ and $L(q)=F(q^2)=K(q^2+q)\, F(q)$.

(2.10) Using the results of § 3 we conclude:

Theorem. *There are homomorphisms K and L from the additive group of A into $M(\mathrm{St}(D_4, A)$ defined by*

$$\begin{aligned} K(a) &= [y_{\varepsilon_1-\varepsilon_2}(1), y_{\varepsilon_1+\varepsilon_2}(a)] \\ L(a) &= [y_{\varepsilon_1-\varepsilon_2}(1), y_{\varepsilon_3-\varepsilon_4}(a^2)] \end{aligned}$$

such that

(i) *$M(\mathrm{St}(D_4, A))$ is the direct product of $K(A)$ and $L(A)$;*

(ii) *the kernel of K is the ideal generated by 2 and all $(q^2+q)(r^2+r)$, $q, r \in A$;*

(iii) *the kernel of L is the ideal generated by all q^2+q, $q \in A$.*

In particular, $M(\mathrm{St}(D_4, A))=1$ if and only if A admits no non-trivial homomorphisms to $\mathbb{F}_2$.

§ 3. The Case $A=\mathbb{F}_2[\varepsilon]$

We still must show that the theorems in (2.6), (2.8), (2.10) hold when $A=\mathbb{F}_2[\varepsilon]$. We begin with (2.10), i.e. type D_4, and deduce from it the results for A_3 and B_3.

(3.1) Let $\Phi = D_4$ and let $\Delta \subset \Phi$ be a simple subsystem. Let $A = \mathbb{F}_2[\varepsilon]$. The maps K and L from A to $M(\mathrm{St}(D_4, A))$ are defined in (2.9); we must show they satisfy (i)–(iii) of Theorem 2.10. The kernel of L contains $\varepsilon = \varepsilon^2 + \varepsilon$ and we know from (2.9) that $L(A)$, $K(A)$ generate $R = M(\mathrm{St}(D_4, A))$. Hence (i)–(iii) will follow once we show that R has (at least) eight elements. Now the surjection $\mathbb{F}_2[\varepsilon] \to \mathbb{F}_2$ sending ε to 0 induces a surjection $M(\mathrm{St}(D_4, A)) \to M(\mathrm{St}(D_4, \mathbb{F}_2))$ whose kernel contains $K(\varepsilon)$. Since $M(\mathrm{St}(D_4, \mathbb{F}_2))$ has four elements ([G1, Table 1]; cf. [S2, (2.4)]), it thus suffices to show:

(3.2) **Theorem.** *$K(\varepsilon) \neq 1$ in $M(\mathrm{St}(D_4, \mathbb{F}_2[\varepsilon]))$.*

The proof of this theorem will be completed in (3.12).

(3.3) Consider the Chevalley group Spin(8) in characteristic 2 (the simply connected group of type D_4). Let L be the points defined over $\mathbb{F}_2$ in the Lie algebra of Spin(8). Then L is a Lie algebra over $\mathbb{F}_2$ containing elements X_α, H_α, $\alpha \in \Phi$ (the set $\{X_\alpha, H_\delta, \alpha \in \Phi, \delta \in \Delta\}$ is a basis for L over $\mathbb{F}_2$). Let Z be the center of L, which is generated by $H_{34} + H_{34'}$ and $H_{12'} + H_{34'}$. The group $G = \mathrm{St}(D_4, \mathbb{F}_2)$ maps onto a subgroup of Spin(8) (isomorphically, but we don't need that). Thus G acts on L via the adjoint representation, and this action leaves Z invariant. Thus $\bar{L} = L/Z$ is a G-module (irreducible, but we won't need that either). We will work with $\bar{L}$ and denote the images of X_α, H_α in $\bar{L}$ by the same letters. Normalize the inner product so that $(\alpha|\alpha) = 2$. Then in $\bar{L}$, $H_\alpha = H_\beta$ if $(\alpha|\gamma) - (\beta|\gamma) \in 2\mathbb{Z}$ for all $\gamma \in \Phi$. So $H_{\pm\varepsilon_i \pm \varepsilon_j} = H_{\varepsilon_i + \varepsilon_j}$, $H_{12} = H_{34}$, and $H_{12'}$, $H_{23'}$ and $\{X_\alpha | \alpha \in \Phi\}$ are a basis for $\bar{L}$.

(3.4) Let $\bar{L}$ act trivially on $\mathbb{Z}/2\mathbb{Z}$. Any bilinear map from $\bar{L}$ to $\mathbb{Z}/2\mathbb{Z}$ is a 2-cocycle with respect to this action. We choose as our cocycle the bilinear map $B\colon \bar{L} \times \bar{L} \to \mathbb{Z}/2\mathbb{Z}$ satisfying

$$B(X_\alpha, X_\beta) = 0 \quad \text{if } \alpha + \beta \neq 0 \text{ or } \alpha < 0.$$

$$B(X_\alpha, X_{-\alpha}) = 1 \quad \text{if } \alpha > 0.$$

$$B(H_\alpha, X_\beta) = B(X_\beta, H_\alpha) = 0.$$

$$B(H_{12'}, H_{12'}) = B(H_{12'}, H_{23'}) = B(H_{23'}, H_{23'}) = 1.$$

$$B(H_{23'}, H_{12'}) = 0.$$

(3.5) *Aside.* The group $H^2(\bar{L}, \mathbb{Z}/2\mathbb{Z})$ is well understood. As a G-module (G acts on $\bar{L}$ and leaves $\mathbb{Z}/2\mathbb{Z}$ alone) it is the degree two part of the symmetric algebra on the dual, $\bar{L}^*$, of $\bar{L}$ (over $\mathbb{F}_2$). The class of B spans the unique one-dimensional invariant subspace. If we knew $H^2(G, \bar{L}^*) = 0$, we could use a spectral sequence to find an element in the Schur multiplier of the semi-direct product of $\bar{L}$ and G [E, § 2]. Since we don't know this, we will apply brute force to split the relevant extension of G by $\bar{L}^*$. (In our exposition we will point out only the splitting, not the extension.) If k is a field of characteristic 2 *larger* than $\mathbb{F}_2$, the same spectral sequence shows that the relevant extension of $\mathrm{St}(D_4, k)$ by its irreducible module of highest weight $\varepsilon_i + \varepsilon_j$ does *not* split (assuming the results of this section for $A = k[\varepsilon]$). So the splitting is exceptional and we are not to blame for the strange looking formulas that describe it.

(3.6) The 2-cocycle B of (3.4) defines a group extension $1 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} M \xrightarrow{p} \bar{L} \to 1$. Let C generate the image of $\mathbb{Z}/2\mathbb{Z}$ in M. We want to write the composition in M multiplicatively because M is not commutative (B is not symmetric). The map i is then given by the formula $i(n) = C^n$, which depends, of course, only on whether n is odd or even. There is a set-theoretic cross-section $s: \bar{L} \to M$ satisfying $C^{B(x,y)} = s(x)\, s(y)\, s(x+y)^{-1}$, $x, y \in \bar{L}$. Since $B(0,0)=0$, we have $s(0)=1$.

(3.7) Set $n_\alpha = s(X_\alpha)$, $d_\alpha = s(H_\alpha)$. We now want to forget B and characterize M in terms of its generators n_α, d_α and C. (The construction of M above can be seen as an existence proof.)

Proposition. *M has a presentation with generators C, n_α, d_α ($\alpha \in \Phi$) subject to the relations:*

$$\begin{aligned}
&[n_\alpha, n_\beta] = 1 \quad \text{if } \alpha+\beta \neq 0,\\
&[n_\alpha m, n_{-\alpha}] = C,\\
&[n_\alpha, d_\beta] = 1,\\
&n_\alpha^2 = 1,\\
&d_\alpha^2 = C,\\
&[d_\alpha, d_\beta] = C^{(\alpha|\beta)},\\
&C^2 = 1,\\
&d_{12} d_{23} = d_{13} C,\\
&d_\alpha = d_\beta \quad \text{if} \quad (\alpha+\beta|\gamma) \in 2\mathbb{Z} \qquad \text{for all } \gamma \in \Phi.
\end{aligned}$$

Proof. First check that these relations hold in M. (Start with the last one; then show that if a relation involves d_α, we may assume $\alpha = \varepsilon_1+\varepsilon_2$, $\varepsilon_2+\varepsilon_3$ or $\varepsilon_1+\varepsilon_3$.) Let M' be the group defined by the presentation. Then C is central in M' and clearly M'/C is isomorphic to the additive group $\bar{L}$. So M' is at most twice as big as $\bar{L}$, and the surjection $M' \to M$ must be a bijection.

(3.8) **Lemma.** *Let (i,j,k,l) be a permutation of $(1,2,3,4)$. Then $d_{\pm\varepsilon_i \pm \varepsilon_j} = d_{\varepsilon_i+\varepsilon_j} = d_{\varepsilon_k+\varepsilon_l}$ and $d_{ij} d_{jk} = d_{ik} C^\sigma C$ where $\sigma=1$ if the permutation is odd and $\sigma=0$ if it is even.*

Proof. The first statement is obvious (cf. (3.3)). We may rewrite the second as

$$d_{ij} d_{jk} d_{ki} = C^\sigma,$$

which clearly holds for the identity permutation. Suppose it holds for a particular choice (i,j,k,l). Then using what we have already proved,

$$\begin{aligned}
d_{ji} d_{ik} d_{kj} &= d_{ij} [d_{ik}, d_{kj}]\, d_{jk} d_{ki}\\
&= d_{ij}\, C\, d_{jk} d_{ki}
\end{aligned}$$

which proves the second statement for (j,i,k,l) as well. Similarly one shows that it holds for (i,k,j,l) and (i,j,l,k) whenever it holds for (i,j,k,l). This completes the proof, since the full permutation group is generated by transpositions of adjacent elements.

(3.9) Next we want to construct an action of $G=\mathrm{St}(D_4,\mathbb{F}_2)$ on M so that $p: M \to \bar{L}$ is equivariant with respect to this action (and the usual action of G on $\bar{L}$). We denote by (i,j,k,l) some permutation of $(1,2,3,4)$. Define

$$C_{rs}=\begin{cases} C & r<s \\ 1 & r>s. \end{cases}$$

For each $\alpha\in\Phi$, define ρ_α as follows:

$$\rho_\alpha(n_\beta)=n_\beta n_{\alpha+\beta} C_{ki} C_{jl} \quad \text{if} \quad \alpha=\pm\varepsilon_i+\varepsilon_j,\ \beta=\pm\varepsilon_j\pm\varepsilon_k, \quad (\alpha|\beta)=-1,$$

$$\rho_\alpha(n_\beta)=n_\beta C \quad \text{if} \quad (\alpha|\beta)=0,$$

$$\rho_\alpha(n_\beta)=n_\beta \quad \text{if} \quad (\alpha|\beta)>0,$$

$$\rho_\alpha(n_{-\alpha})=n_{-\alpha} d_\alpha n_\alpha,$$

$$\rho_\alpha(C)=C,$$

$$\rho_\alpha(d_\gamma)=d_\gamma n_\alpha^{(\alpha|\gamma)} C^{(\beta|\gamma)} \quad \text{if} \quad \alpha=\pm\varepsilon_i\pm\varepsilon_j, \quad \beta=+\varepsilon_j+\varepsilon_k, \quad \text{and} \quad i<j,\ k<l.$$

(Note that β is determined by α.)

In particular, $\rho_\alpha(d_\gamma)=d_\gamma n_\alpha C_{pq} C_{rs}$ if $\alpha=\pm\varepsilon_p\pm\varepsilon_q$, $\gamma=+\varepsilon_q+\varepsilon_r$, and $\rho_\alpha(d_\alpha)=d_\alpha C$, if (p,q,r,s) is a permutation of $(1,2,3,4)$.

We claim that this list defines ρ_α as an element of $\mathrm{Aut}(M)$, the automorphism group of M. To check this, we must show that ρ_α preserves the defining relations of M (Proposition (3.7)). We also claim that the automorphism of $\bar{L}$ induced by ρ_α is the same as the action of $x_\alpha(1)$. Finally we claim that $\{\rho_\alpha|\alpha\in\Phi\}$ satisfy the defining relations of G so that $x_\alpha(1)\mapsto\rho_\alpha$ defines a homomorphism $G\to\mathrm{Aut}(M)$ (in (3.5) we referred to this homomorphism as a "splitting"). So, for instance, we claim that $\rho_\alpha\rho_\beta=\rho_{\alpha+\beta}\rho_\beta\rho_\alpha$ when $(\alpha|\beta)=-1$. This is proved by evaluating both sides on the generators of M. Since the analogous relation is valid in $\bar{L}$, all that might go wrong is that the powers of C which are produced in this way don't match. Before the powers of C can be compared, however, one first must see that all the other factors cancel. This may involve a rearrangement of the other factors, which produces a C, e.g. $n_{-\beta}n_\beta=n_\beta n_{-\beta}C$. It may also involve combining factors as in Lemma (3.8), which may also produce a C.

(3.10) *Example.* Take $\alpha=\varepsilon_i+\varepsilon_j$, $\beta=-\varepsilon_j+\varepsilon_k$, $\gamma=-\varepsilon_i-\varepsilon_k$. Then $\rho_\alpha\rho_\beta(n_\gamma)=n_\gamma n_{\alpha+\gamma} C_{kj} C_{il} n_{\beta+\gamma} d_\alpha n_\alpha C_{ij} C_{kl}$, and $\rho_{\alpha+\beta}\rho_\beta\rho_\alpha(n_\gamma)=n_\gamma d_{\alpha+\beta} n_{\alpha+\beta} n_{\beta+\gamma} n_\beta C_{jk} \cdot C_{il} C_{ij} C_{kl} n_{\alpha+\gamma} n_\alpha C_{ji} C_{kl} d_\beta n_{\alpha+\beta} C_{ik} C_{jl} n_\beta C_{kj} C_{il}$. By Lemma (3.8), $d_{\alpha+\beta}d_\beta=d_\alpha C^\sigma C$, where $\sigma=0$ if (i,k,j,l) is an even permutation and $\sigma=1$ otherwise. Thus in order that $\rho_\alpha\rho_\beta(n_\gamma)=\rho_{\alpha+\beta}\rho_\beta\rho_\alpha(n_\gamma)$, we must have

$$1=C^\sigma C_{jk} C_{il} C_{kl} C_{ji} C_{ik} C_{jl}$$

or

$$C^\sigma=C_{jk} C_{ji} C_{jl} C_{ik} C_{kl} C_{il}.$$

But this last term on the right is C^τ, where τ is the length of the permutation (l,k,i,j). (The length counts the number of roots that are made negative, or the number of transpositions of neighbors necessary to build the permutation.) Clearly (l,k,i,j) and (i,k,j,l) have the same signature, so that $C^\sigma=C^\tau$ indeed.

(3.11) Example (3.10) deals with the worst case. We leave to the reader the task of checking that $\rho_\alpha^2=1$, $\rho_\alpha\rho_\gamma=\rho_\gamma\rho_\alpha$ if $(\alpha|\gamma)=0$ and the remaining cases of $\rho_\alpha\rho_\beta=\rho_{\alpha+\beta}\rho_\beta\rho_\alpha$ if $(\alpha|\beta)=-1$. These relations suffice to define the homomorphism $G\to \mathrm{Aut}(M)$ (cf. (3.9)), since the relation $[\rho_\alpha,\rho_{\alpha+\beta}]=1$ if $(\alpha|\beta)=-1$ follows from them:

$$\begin{aligned}1&=[\rho_\alpha^2,\rho_\beta]\\&={}^{\rho_\alpha}[\rho_\alpha,\rho_\beta][\rho_\alpha,\rho_\beta]\\&=\rho_\alpha\cdot\rho_{\alpha+\beta}\cdot\rho_\alpha^{-1}\cdot\rho_{\alpha+\beta}\\&=[\rho_\alpha,\rho_{\alpha+\beta}].\end{aligned}$$

(3.12) Now let $H=M\rtimes G$, the semi-direct product of M and G. Since $p\colon M\to\bar{L}$ is equivariant with respect to the actions of G on M and $\bar{L}$, there is an induced surjective homomorphism $H\to\bar{L}\rtimes G$ with kernel generated by C. In particular, C is central in H. There is also a surjection $\mathrm{St}(D_4,\mathbb{F}_2[\varepsilon])\to\bar{L}\rtimes G$ defined by $x_\alpha(1)\mapsto x_\alpha(1)$, $x_\alpha(\varepsilon)\mapsto X_\alpha$ which induces a homomorphism $M(\mathrm{St}(D_4,\mathbb{F}_2[\varepsilon]))\to H$. The image of $[y_\alpha(1), y_\beta(\varepsilon)]$ is the commutator of $x_\alpha(1)$ and n_β in H by the central trick. So if $(\alpha|\beta)=0$, the image of $[y_\alpha(1), y_\beta(\varepsilon)]$ in H is $\rho_\alpha(n_\beta)\,n_\beta^{-1}=C\neq 0$. In particular, $K(\varepsilon)$ has non-trivial image in H, proving Theorem (3.2) and, therefore, Theorem (2.10).

(3.13) A homomorphism $\mathrm{St}(B_3,\mathbb{F}_2[\varepsilon])\to\mathrm{St}(D_4,\mathbb{F}_2[\varepsilon])$ may be defined by $x_{s\varepsilon_j}(t)\mapsto x_{s\varepsilon_j+\varepsilon_4}(t)\,x_{s\varepsilon_j-\varepsilon_4}(t)$ for $s=\pm1$, $x_\alpha(t)\mapsto x_\alpha(t)$ if α is long (e.g. $\alpha=\varepsilon_1+\varepsilon_3$). This induces a homomorphism $M(\mathrm{St}(B_3,\mathbb{F}_2[\varepsilon]))\to M(\mathrm{St}(D_4,\mathbb{F}_2[\varepsilon]))$ sending $[y_{12'}(1), y_{12}(r)]$ to $[y_{12'}(1), y_{12}(r)]$. Thus it follows from (2.7) and (2.10) that Theorem (2.8) holds for $A=\mathbb{F}_2[\varepsilon]$, which completes the proof of that theorem.

(3.14) There is also a homomorphism $\mathrm{St}(A_3,\mathbb{F}_2[\varepsilon])\to\mathrm{St}(D_4,\mathbb{F}_2[\varepsilon])$ induced by the inclusion of Dynkin diagrams $A_3\subset D_4$ ($\{\alpha_1,\alpha_2,\alpha_3\}\subset\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$). We use it as in (3.13) to complete the proof of Theorem (2.6).

§ 4. Applications

(4.1) There is a natural homomorphism $\mathrm{St}(\Phi,A)\to G(\Phi,A)$, sending $x_\alpha(t)$ to $e_\alpha(t)$ [S3, § 1A]. Its kernel is denoted $K_2(\Phi,A)$, its image by $E(\Phi,A)$, and $K_1(\Phi,A)$ is the pointed set $G(\Phi,A)/E(\Phi,A)$ which is its cokernel.

(4.2) Whenever $K_2(\Phi,A)$ is central in $\mathrm{St}(\Phi,A)$, there is an exact sequence [S2, Proposition 1.1]

$$0\to M(\mathrm{St}(\Phi,A))\to M(E(\Phi,A))\to K_2(\Phi,A)\to 0.$$

If $\mathrm{rk}\,\Phi\geqq 3$, we have computed the kernel in this sequence and have seen that it depends only on what homomorphisms exist from A to $\mathbb{F}_3$ and $\mathbb{F}_2[\varepsilon]$. Thus whenever $K_2(\Phi,A)$ is central and computable, we can compute the Schur multiplier of the subgroup $E(\Phi,A)$.

(4.3) *Examples.* If $\Phi=\mathrm{A}_3$, $K_2(\Phi,A)$ is central in $\mathrm{St}(\Phi,A)=\mathrm{St}(4,A)$ for any *commutative* ring A (van der Kallen, to appear). In general if A satisfies Bass' stable range condition SR_m [S3, § 1C] for small enough m, $\mathrm{K}_2(\Phi,A)$ is central in $\mathrm{St}(\Phi, A)$ [S3, Corollary (3.4)]. Here is a list of sufficient conditions on m.

$$\Phi = B_3, \quad m=2,$$
$$\Phi = C_3, \quad m=2,$$
$$\Phi = D_4, \quad m \leqq 3,$$
$$\Phi = F_4, \quad m \leqq 3.$$

(4.4) If $K_1(\Phi, \mathrm{A}) = 1$, we may replace $E(\Phi, A)$ by the Chevalley group $G(\Phi, A)$ in (4.2).

Examples. 1. If A is semi-local, $K_1(\Phi, A) = 1$ [Ma, Cor. 4.4b; S3, Theorem 2.2 and Cor. 2.3].

2. If A is the ring of integers in an algebraic number field, $K_1(\Phi, A) = 1$ if $\mathrm{rk}\,\Phi \geqq 2$ [Ma, Cor. 4.6; B-M-S, Cor. 4.3a and Cor. 12.5].

(4.5) If the maximal spectrum of A has dimension $\leqq 1$, $K_2(A_3, A) = K_2(A)$ [vdK3]. Hence:

Proposition. *If A is the ring of integers in an algebraic number field, there is an exact sequence*

$$0 \to M(\mathrm{St}(4, A)) \to M(SL(4, A)) \to K_2(A) \to 0.$$

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