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A new interpretation for the mass of a classical relativistic particle

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Abstract

Based on a recent classification of coadjoint orbits of the full Poincaré group, we give a new group theoretic interpretation for the mass of a classical relativistic particle.

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In the late nineteen sixties there was some work on the classification of coadjoint orbits of the Poincaré group done in [7] using Wigner's little group method. The thesis concentrated on geometric quantization, as did Rawnsley [6], who considered a general semidirect product of a Lie group and a vector space.

Using the recent classification of all the coadjoint orbits of the Poincaré group in [4], we obtain a new interpretation for the mass of a classical relativistic particle [1,5,8].

We interpret nonzero mass of a classical relativistic particle to be a nonzero *modulus* for the coadjoint orbit of the full Poincaré group which represents the particle. To explain what we mean by a modulus, consider the group $G = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{R}\}$. Since G is abelian, $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right)$ in G if and only if $a = a'$. Thus the conjugacy classes of elements of G are parametrized by the real number a, which is clearly not an eigenvalue of any element of G. For the full Poincaré group our modulus is independent of the component of the coadjoint orbit. In our classification it is consistent to take it positive, and we did so. Since mass has been measured to be positive, we interpret mass as being the modulus, not its negative. It follows that in our interpretation there are *no* classical relativistic particles with *negative* mass. Thus we partly resolve a difficulty which plagued Souriau. (See the last footnote on page 192 of [8].) The Lorentz length squared of the energy-momentum vector of the particle is still the square of the mass.

The components of the coadjoint orbit of a classical relativistic particle are labeled by physical data which characterize the particle; namely, the sign of the spin for a massive classical relativistic particle with spin; the sign of the energy for a massive classical relativistic particle without spin; the sign of the energy and the helicity for a massless classical relativistic particle.

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Let us recall what mass means in the standard classification of classical relativistic particles. In [3] an elementary system = (classical relativistic particle) is the representation space (which is a specific Hilbert space) of a single irreducible unitary representation of the full Poincaré group. Such a representation is characterized by mass squared (m^2) and spin ($j(j+1)$) or helicity. There is a list in [3] of irreducible unitary representations of the universal covering group of the identity component of the Poincaré group. A complete list of the irreducible unitary representations of the *full Poincaré group* do not seem to be known, in spite of the "folklore" in the physics community that they are, [2,9]. In [8] an elementary system is a connected component of a coadjoint orbit of the restricted Poincaré group (= the identity component of the full Poincaré group). The coadjoint orbits of the restricted Poincaré group are classified by mass (= the signed square roots of the Lorentz length of the energy momentum vector), spin or helicity, [8]. The relation between the representation theoretic and coadjoint orbit definitions of an elementary system is the procedure of geometric quantization, see [6,8], which constructs an irreducible unitary representation starting from a coadjoint orbit.

1. Facts about the Poincaré group

First we recall some standard facts about the Poincaré group, which we will use later on. Let $\{e_1, e_2, e_3, e_4\}$ be a basis of Minkowski space $(\mathbb{R}^4, \gamma)$ such that the Gram matrix of the Lorentz inner product γ is $G = \mathrm{diag}(-I_3, 1)$, that is, γ is diagonal with signature $---+$. The Lorentz group $\mathrm{O}(3,1)$ is the Lie group of linear maps of $\mathbb{R}^4$ into itself which preserve γ. The Lie algebra $\mathrm{o}(3,1)$ of $\mathrm{O}(3,1)$ consists of 4×4 real matrices M such that $M^T G + GM = 0$, that is, $M = \begin{pmatrix} \hat{\ell} & g \\ g^T & 0 \end{pmatrix}$, where

$$\ell = (\ell_1, \ell_2, \ell_3), \quad g \in \mathbb{R}^3 = \mathrm{span}\{e_1, e_2, e_3\} \quad \text{and} \quad \hat{\ell} = \begin{pmatrix} 0 & -\ell_3 & \ell_2 \\ \ell_3 & 0 & -\ell_1 \\ -\ell_2 & \ell_1 & 0 \end{pmatrix}.$$

The Poincaré group $\mathcal{P}$ is the group of *affine* linear Lorentz transformations of Minkowski space into itself, that is, $\mathcal{P}$ is the semidirect product of the Lorentz group and the abelian group $\mathbb{R}^4$. We can write $(S, C) \in \mathcal{P} \subseteq \mathrm{O}(3,1) \times \mathbb{R}^4$ as the 5×5 real matrix $\begin{pmatrix} S & C \\ 0 & 1 \end{pmatrix}$. Group multiplication $\cdot$ in $\mathcal{P}$ is then given by matrix multiplication. An element (Σ, Γ) of the Lie algebra $\wp \subseteq \mathrm{o}(3,1) \times \mathbb{R}^4$ of the Poincaré group can be written as the 5×5 real matrix $\begin{pmatrix} \Sigma & \Gamma \\ 0 & 0 \end{pmatrix}$.

Using the nondegenerate inner product

$$\langle | \rangle : \wp \times \wp \to \mathbb{R} : \big((M, P), (\Sigma, \Gamma)\big) \mapsto -\frac{1}{2} \operatorname{tr} M\Sigma - \gamma(P, \Gamma), \tag{1}$$

we may identify $\wp$ with its dual space $\wp^*$. Thus we think of (M, P) as an element ν of $\wp^*$. Under this identification, the coadjoint action

$$\mathcal{P} \times \wp^* \to \wp^* : \big((S, C), \nu\big) \mapsto \mathrm{Ad}^T_{(S,C)^{-1}} \nu$$

of $\mathcal{P}$ on $\wp^*$ becomes the action

$$\widetilde{\mathrm{Ad}} : \mathcal{P} \times \wp \to \wp : \big((S, C), (M, P)\big) \mapsto (SMS^{-1} + L_{C,SP}, SP). \tag{2}$$

$L_{C,SP}$ is the linear map of $\mathbb{R}^4$ into itself defined by

$$L_{C,SP}\Gamma = \gamma(SP, \Gamma)C - \gamma(C, \Gamma)SP.$$

Note that $L_{C,SP} \in \mathrm{o}(3,1)$. The energy-momentum vector associated to (M, P) is $P = \begin{pmatrix} p \\ E \end{pmatrix}$ and its polarization vector is $W = \begin{pmatrix} p \times g + E\ell \\ \langle p, \ell \rangle \end{pmatrix}$. Here $\langle , \rangle$ is the standard Euclidean inner product on $\mathbb{R}^3$. It is known that the Casimir functions $C_1 : \wp \to \mathbb{R} : (M, P) \mapsto \gamma(P, P)$ and $C_2 : \wp \to \mathbb{R} : (M, P) \mapsto \gamma(W, W)$ are invariant under the action $\widetilde{\mathrm{Ad}}$.

Minkowski space $(\mathbb{R}^4, \gamma)$ has two standard involutions: space inversion

$$I_s : \mathbb{R}^4 \to \mathbb{R}^4 : \begin{pmatrix} r \\ t \end{pmatrix} \mapsto \begin{pmatrix} -r \\ t \end{pmatrix}$$

and time reversal

$$I_t : \mathbb{R}^4 \to \mathbb{R}^4 : \begin{pmatrix} r \\ t \end{pmatrix} \begin{pmatrix} r \\ -t \end{pmatrix}.$$

These involutions are Lorentz transformations that do not lie in the identity component of the Lorentz group. Therefore $(I_s, 0)$ and $(I_t, 0)$ are elements of the Poincaré group which do not lie in its identity component $\mathcal{P}^\circ$. In fact, the group of connected components $\mathcal{P}/\mathcal{P}^\circ$ of $\mathcal{P}$ is $\{e, I_s, I_t, I_s \circ I_t\}$. A calculation shows that

$$(M_s, P_s) = \widetilde{\mathrm{Ad}}_{(I_s,0)}(M, P) = \left(\begin{pmatrix} \hat{\ell} & -g \\ -g^T & 0 \end{pmatrix}, \begin{pmatrix} -p \\ E \end{pmatrix} \right)$$

and

$$(M_t, P_t) = \widetilde{\mathrm{Ad}}_{(I_t,0)}(M, P) = \left(\begin{pmatrix} \hat{\ell} & -g \\ -g^T & 0 \end{pmatrix}, \begin{pmatrix} p \\ -E \end{pmatrix} \right).$$

Therefore the polarization vector W_s corresponding to (M_s, P_s) is $\left(\begin{smallmatrix} p\times g + E\,\ell \\ -\langle \ell, p\rangle \end{smallmatrix}\right)$; whereas the polarization vector W_t corresponding to (M_t, P_t) is $\left(\begin{smallmatrix} -p\times g - E\ell \\ \langle \ell, p\rangle \end{smallmatrix}\right)$.

2. The classification

From the classification of the coadjoint orbits of the Poincaré group given in [4], we extract the following physically significant coadjoint orbits, following the criteria in [8].

1. Let $\mathcal{O}$ be the coadjoint orbit containing the covector

$$\nu = (M^\circ, P^\circ) = \left(\begin{pmatrix} \beta\hat{e}_3 & 0 \\ 0 & 0 \end{pmatrix}, \mu e_4 \right),$$

where $\mu > 0$ is a modulus. The polarization vector associated to ν is $W^\circ = \mu\beta e_3$. The value of the Casimir C_1 at ν is μ^2 and the value of C_2 at ν is $-\mu^2\beta^2$. Therefore the classical relativistic particle corresponding to $\mathcal{O}$ has positive mass μ and spin $\pm\beta = \pm\sqrt{\frac{-\gamma(W^\circ, W^\circ)}{\gamma(P^\circ, P^\circ)}}$. At ν the isotropy group

$$\mathcal{P}_{(M^\circ,P^\circ)} = \{(S, C) \in \mathcal{P} \mid \widetilde{\mathrm{Ad}}_{(S,C)}(M^\circ, P^\circ) = (M^\circ, P^\circ)\},$$

given by

$$\left\{ (S, C), (I_s, 0)\cdot(S, C) \in \mathcal{P} \mid (S, C) = \left(\begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 & 0 \\ \sin\vartheta & \cos\vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \lambda e_4 \right) \right\}$$

with $\vartheta, \lambda \in \mathbb{R}$, has two connected components. Thus the coadjoint orbit $\mathcal{O}$ has two connected components $\mathcal{O}^\pm$, which are interchanged under time reversal. In other words, if we assume that $(M^\circ, P^\circ) \in \mathcal{O}^+$, then $(M_t^\circ, P_t^\circ) \in \mathcal{O}^-$ and conversely. Assuming that the value of spin is β on $\mathcal{O}^+$, the value of spin on $\mathcal{O}^-$ is $-\beta$, since

$$W_t^\circ = -W^\circ = \mu(-\beta)e_3.$$

2. Let $\mathcal{O}$ be the coadjoint orbit containing $\nu = (M^\circ, P^\circ) = (0, \mu e_4)$, where $\mu > 0$ is a modulus. The polarization vector corresponding to ν is $W^\circ = 0$. The value of the Casimir C_1 at ν is μ^2 and the value of C_2 at ν is 0. Therefore the classical relativistic particle corresponding to $\mathcal{O}$ has positive mass μ and no spin. At ν the isotropy group

$$\mathcal{P}_{(M^\circ,P^\circ)} = \left\{ (S, C), (I_s, 0)\cdot(S, C) \in \mathcal{P} \mid (S, C) = \left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \lambda e_4 \right) \right\}$$

with $A \in \mathrm{SO}(3, \mathbb{R})$ and $\lambda \in \mathbb{R}$ has two connected components. Therefore the coadjoint orbit $\mathcal{O}$ has two connected components $\mathcal{O}^\pm$, which are interchanged by time reversal I_t. Assuming that $(M^\circ, P^\circ) \in \mathcal{O}^+$, then $(M_t^\circ, P_t^\circ) \in \mathcal{O}^-$ and conversely. Assuming that energy E is positive on $\mathcal{O}^+$, then it is negative on $\mathcal{O}^-$, because $P_t^\circ = -P^\circ = -\mu e_4$. Consequently, the classical relativistic particle corresponding to the coadjoint orbit $\mathcal{O}$ has positive mass μ, no spin, and positive or negative energy.

3. Let $\mathcal{O}$ be the coadjoint orbit containing

$$\nu = (M^\circ, P^\circ) = \left(\begin{pmatrix} \beta \hat{e}_1 & 0 \\ 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}(e_1 + e_4) \right),$$

where $\beta > 0$. The polarization vector corresponding to ν is $W^\circ = \frac{\beta}{\sqrt{2}}(e_1 + e_4) = \beta P^\circ$. The value of the Casimir C_1 at ν is $\gamma(\frac{1}{\sqrt{2}}(e_1 + e_4), \frac{1}{\sqrt{2}}(e_1 + e_4)) = \frac{1}{2} - \frac{1}{2} = 0$ and the value of the Casimir C_2 at ν is 0. Thus the classical relativistic particle corresponding to $\mathcal{O}$ has no mass, but its spin is β. A calculation shows that $\mathcal{O}$ has four connected components $\mathcal{O}^{\pm,\pm}$. Space inversion I_s interchanges $\mathcal{O}^{+,+}$ and $\mathcal{O}^{-,+}$. Time inversion I_t interchanges $\mathcal{O}^{+,+}$ and $\mathcal{O}^{+,-}$. Thus space time inversion $I_s \circ I_t$ interchanges $\mathcal{O}^{+,+}$ and $\mathcal{O}^{-,-}$. Assuming that $(M^\circ, P^\circ) \in \mathcal{O}^{+,+}$, then time reversal I_t changes the sign of the energy, since $P_t^\circ = \frac{1}{\sqrt{2}}(e_1 - e_4)$. Thus the second $\pm$ sign in $\mathcal{O}^{+,+}$ is the sign of the energy. Since $W_s^\circ = \frac{\beta}{\sqrt{2}}(e_1 - e_4) = -\beta P_s^\circ$, space reversal changes the direction of the polarization vector relative to the energy-momentum vector. Thus the first sign in $\mathcal{O}^{+,+}$ is the helicity. Consequently, $\mathcal{O}$ corresponds to a classical relativistic particle with no mass, but with positive spin, positive or negative energy, and positive or negative helicity.

3. Normal form

Since our classification of classical nonrelativistic particles depends on the existence of a new invariant of the coadjoint action of the Poincaré group, it behooves us to give a method for calculating it. We use ideas from [4] but give a slightly different algorithm, which uses little cotypes rather than affine cotypes, to determine normal forms for the coadjoint orbits listed in Section 2.

Suppose that we are given $(M, P) \in \wp \subseteq \mathrm{o}(3,1) \times \mathbb{R}^4$. To find the normal form, we perform the following steps.

1. Check if $P \neq 0$ and $\gamma(P, P) \neq 0$. Set $\gamma(P, P) = \varepsilon \mu^2$ with $\varepsilon^2 = 1$ and $\mu > 0$. Then μ is a modulus. Let $W = \mathrm{span}\{P\}^\gamma$ be the γ-orthogonal complement in $\mathbb{R}^4$ of the subspace spanned by P. There are two cases.

(a) Check if $\varepsilon = 1$ and the Gram matrix of $\gamma|W$ is $-I_3$. Check if the characteristic polynomial of $Y = M|W$ is $\lambda(\lambda^2 + \beta^2)$ for some $\beta > 0$. Let P_0 be the normalized eigenvector of Y corresponding to the eigenvalue 0. Let P_1 be a normalized vector such that on $\mathrm{span}\{P_1, \beta^{-1} Y P_1\}$ we have $Y^2 + \beta^2 = 0$. Set $P_3 = \mu^{-1} P$. Then $\{P_1, \beta^{-1} Y P_1, P_0, P_3\}$ is a γ-orthogonal basis of $\mathbb{R}^4$ with respect to which the Gram matrix of γ is $\mathrm{diag}(-I_3, 1)$ and the pair (M, P) has the normal form of case 1 of Section 2.

(b) Check if the characteristic polynomial of $Y = M|W$ is λ^3 and Y is diagonalizable. Let P_0, P_1, and P_2 be an orthonormal system of eigenvectors of Y. Set $P_3 = \mu^{-1} P$. Then $\{P_0, P_1, P_2, P_3\}$ is a γ-orthogonal basis of $\mathbb{R}^4$ with respect to which the Gram matrix of γ is $\mathrm{diag}(-I_3, 1)$ and the pair (M, P) has the normal form of case 2 of Section 2.

2. Check if $P \neq 0$ and $\gamma(P, P) = 0$. Find a nonzero vector $\widehat{P}$ such that $\gamma(P, \widehat{P}) = 1$ and $\gamma(\widehat{P}, \widehat{P}) = 0$. Let $W = \mathrm{span}\{\widehat{P}, P\}$. The Gram matrix of $\gamma|W$ is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. On W^γ, there is a basis $\{Q, Q'\}$ such that the Gram matrix of $\gamma|W^\gamma$ is $-I_2$. With respect to the basis $\{f_1, f_2, f_3, f_4\} = \{\widehat{P}, Q, Q', P\}$ the Gram matrix of γ is

$$G' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and the pair (M, P) has the form

$$(M', P') = \left(\begin{pmatrix} a & -x^T G' & 0 \\ y & Y' & x \\ 0 & -y^T G' & -a \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right). \tag{3}$$

Here $x, y \in \mathrm{span}\{f_2, f_3\}$, $a \in \mathbb{R}$, and $(Y')^T + Y' = 0$. Change coordinates using

$$\left(I_4, \begin{pmatrix} -a \\ -y \\ 0 \end{pmatrix} \right) \in \mathcal{P}.$$

From (2) we see that (M', P') becomes

$$(\widetilde{M}, \widetilde{P}) = \left(\begin{pmatrix} 0 & -x^T G' & 0 \\ 0 & Y' & x \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right). \tag{4}$$

Look at the pair $(Y' = \widetilde{M}|W^\gamma, x)$, where $x \in W^\gamma$. Check if $x = 0$ and $Y' \neq 0$. Then its characteristic polynomial on W^γ is $\lambda^2 + \beta^2$ for some $\beta > 0$. There is no modulus. With respect to the basis

$$\left\{ \frac{1}{\sqrt{2}}(f_1 - f_4), f_2, \beta^{-1} Y' f_2, \frac{1}{\sqrt{2}}(f_1 + f_4) \right\}$$

the Gram matrix of γ is $\mathrm{diag}(-I_3, 1)$ and the matrix of the pair $(\widetilde{M}, \widetilde{P})$ has the normal form of case 3 of Section 2.

To relate the coadjoint orbits in Section 2 to the classification in [4], we note that cases 1, 2, and 3 are denoted there by $\nabla_3^+(0)$, $\mu + \Delta_0^-(i\beta, IP) + \Delta_0^-(0)$, $\nabla_3^+(0)$, $\mu + \Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^-(0)$, and $\nabla_4(0, 0) + \Delta_0^-(i\beta, IP)$, respectively.

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