THE MERKURJEV-SUSLIN THEOREM

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SUMMARY. This paper reflects a talk given at the conference. We explain some of the ingredients of Merkurjev's "elementary" proof of the Merkurjev-Suslin theorem [14].

§1. THE MAIN THEOREM

Let F be an arbitrary (commutative) field, n a positive integer with non-zero image in F. The K-theory of F forms a graded anti-commutative ring $\bigoplus_{i\geq 0} K_i(F)$ with $K_0(F) \cong \mathbb{Z}$, $K_1(F) \cong F^* = GL(1,F)$ [16], [27].
We write $\{a\}$ for the element of $K_1(F)$ that corresponds with $a \in F^*$, so that $\{a\} + \{b\} = \{ab\}$. Multiplying $\{a\},\{b\} \in K_1(F)$ yields an element of $K_2(F)$ that is written as $\{a,b\}$. One calls $\{a,b\}$ a Steinberg symbol. It may also be described in terms of Schur multipliers and matrices [16].

Matsumoto's Theorem [16]. The homomorphism of additive groups $K_1(F) \otimes_{\mathbb{Z}} K_1(F) \to K_2(F)$ that sends $\{a\} \otimes \{b\}$ to $\{a,b\}$ is surjective and its kernel is generated by the $\{a\} \otimes \{1-a\}$ with both a and 1-a invertible.

The same theorem holds true for a (commutative) local ring whose residue field has more than six elements [12]. Matsumoto's theorem provides a presentation for $K_2(F)$ and therefore also for $k_2(F) = K_2(F)/nK_2(F)$. Thus this group is in a sense known.
We will give two descriptions of the main theorem of Merkurjev and Suslin, one with Galois cohomology, one with Brauer groups. Let μ_n denote the group of n-th roots of unity in an algebraic closure $\overline{F}$ of F and consider μ_n as a module for the absolute Galois group $Gal(F^s:F)$, where F^s is the separable closure of F in $\overline{F}$. By the Hilbert 90 theorem the Galois cohomology group $H^1(F,\mu_n)$ is isomorphic with $K_1(F)/nK_1(F)$. We now exploit the product structures in K-theory and in Galois cohomology to get up from degree 1 to degree 2. Writing $\mu_n^{\otimes 2}$ for $\mu_n \otimes_{\mathbb{Z}} \mu_n$ (with diagonal action of the Galois group) we have a cup product

$$H^1(F,\mu_n) \otimes_{\mathbb{Z}} H^1(F,\mu_n) \to H^2(F,\mu_n^{\otimes 2})$$

and therefore a homomorphism

$$(K_1(F)/nK_1(F)) \otimes_{\mathbb{Z}} (K_1(F)/nK_1(F)) \to H^2(F,\mu_n^{\otimes 2})$$

Using the Matsumoto theorem one shows that this homomorphism factors through

$k_2(F) = K_2(F)/nK_2(F)$ and the resulting homomorphism

$$\alpha_{F,n}: k_2(F) \to H^2(F,\mu_n^{\otimes 2})$$

is called the Galois symbol or the norm residue homomorphism. The Merkurjev-Suslin theorem now simply reads

MAIN THEOREM OF MERKURJEV AND SUSLIN [15].

For all F,n as above, $\alpha_{F,n}$ is an isomorphism.

As will be explained later, it is not difficult to see that the crucial case in the Main Theorem is the case where $\mu_n \subset F$ so that $\mu_n^{\otimes 2}$ is simply a cyclic group of order n on which the Galois group acts trivially. In this situation, choose a primitive n-th root of unity ω in $\mu_n = H^0(F,\mu_n)$ and use cup product by ω to identify $H^2(F,\mu_n^{\otimes 2})$ with $H^2(F,\mu_n) \cong {}_nBr(F)$, the n-torsion subgroup of the Brauer group. (If A is an abelian group, ${}_nA$ denotes the kernel of multiplication by n: $A \to A$). One finds that

$$\alpha_{F,n}: k_2(F) \to {}_nBr(F)$$

sends the coset of the Steinberg symbol $\{a,b\}$ to the similarity class of the cyclic algebra $A_\omega(a,b)$, where $A_\omega(a,b)$ is the central simple F algebra of dimension n^2, with linear basis $\{x^i y^j \mid 0 \leq i < n,\ 0 \leq j < n\}$ and algebra generators x, y, so that $xy = \omega yz$, $x^n = a$, $y^n = b$ [16]. Thus the following corollary expresses the surjectivity of $\alpha_{F,n}$ and gives some idea of the force of the main theorem.

COROLLARY. Let $\mu_n \subset F$ and let A be a central simple F algebra with $[A] \in {}_nBr(F)$. Then A is similar to a tensor product of cyclic algebras $A_\omega(a_i,b_i)$, $a_i,b_i \in F^*$.

Even for n = 2 this settles an old problem in the theory of Brauer groups. The case n = 2 is actually Merkurjev's theorem.

MERKURJEV's THEOREM [3], [13], [25]. If F has characteristic different from 2, then $\alpha_{F,2}$ is an isomorphism.

For a discussion of how the situation was before this breakthrough, we refer to [2], [25]. The main theorem and some of the auxiliary results obtained in its proof (Hilbert 90 and reduced norm for K_2,...) have had consequences in several areas. Apart from K-theory, Brauer groups and quadratic forms, let us mention L-theory (surgery groups) and Chow groups (intersection of cycles on a rational surface say).

§2. SEVERI-BRAUER VARIETIES

One of the ingredients in the proof is the study of cohomology of sheaves of K-groups on Severi-Brauer varieties. Therefore we now turn to these varieties, starting with a simple example.

EXAMPLE. Let $F = \mathbb{Q}$, $n = 2$ and consider the conic $1 + Y^2 = 3Z^2$. It is somewhat better to pass to the projective plane where our conic C is given by $X^2 + Y^2 = 3Z^2$ in homogeneous coordinates. Now recall that a conic is isomorphic with a projective line via the following construction.
Fix a point P on the conic and assign to a point Q on the conic the line PQ. This gives a 1-1 correspondence between the points on the conic C and the lines through P, hence between C and the projective line $\mathbb{P}^1$. Or does it? The conic C of our example is actually "empty". There are no rational solutions of $1 + Y^2 = 3Z^2$. (To see this one may look at the 2-adic completion of $\mathbb{Q}$.) We write $C(\mathbb{Q}) = \emptyset$, where, for a field E containing $F = \mathbb{Q}$ one denotes by C(E) the set of E rational points on C (= points with coordinates in E). Over $\mathbb{Q}$ we can not choose P on the conic. But if we extend scalars suitably, then it all works: If E is a field containing $\mathbb{Q}$ and such that there is $P \in C(E)$, then the lines through P defined over E (i.e. with slopes in E) corresponds exactly with elements of C(E) and the isomorphism between the conic and $\mathbb{P}^1$ is defined over E. If we write C_E for what one gets from C by extension of scalars $F \to E$ then we may summarize

$$C_E \simeq \mathbb{P}^1_E \Leftrightarrow C(E) \neq \emptyset.$$

To be precise one should view C_E as a scheme over E, viz. the projective spectrum $\mathrm{Proj}(E[X,Y,Z]/(X^2+Y^2-3Z^2))$. [9]
This scheme is smooth over E. We say that E splits C if $C_E \simeq \mathbb{P}^1_E$. For example, $\mathbb{Q}(i)$ splits C, because $1 + i^2 = 3.0^2$. The conic C is closely related with the cyclic algebra $A_{-1}(-1,3)$. Over $\mathbb{Q}(i)$ this algebra is simply the matrix algebra $M_2(\mathbb{Q}(i))$ with algebra generators $x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $y = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ satisfying the required relations $x^2 = -1$, $y^2 = 3$, $xy = -yx$. Thus $\mathbb{Q}(i)$ is a splitting field [1] of $A_{-1}(-1,3)$.
For $p,q,r,s \in \overline{F}$ the reduced norm $\mathrm{Nrd}(p+qx+ry+sxy)$ of $p + qx + ry + sxy$ is just the determinant $p^2 + q^2 - 3r^2 + 3s^2$ of the corresponding matrix in $M_2(\overline{F}) = \overline{F} \otimes_F A_{-1}(-1,3)$. Elements of reduced norm zero are zero divisors and conversely. Consider a field extension E of $F = \mathbb{Q}$. One sees:
E splits $A_{-1}(-1,3) \Leftrightarrow$
There is a two dimensional E linear subspace of zero divisors in $E \otimes_F A_{-1}(-1,3) \Leftrightarrow$
There is a non-zero solution of $p^2 + q^2 - 3r^2 = 0$ over E $\Leftrightarrow$
The "Severi-Brauer variety" C has an E rational point $\Leftrightarrow$
$C_E \cong \mathbb{P}^1_E$.

Thus E splits the algebra if and only if it splits the conic.

This example generalizes as follows [4]. Let A be a central simple algebra over the field F, $\dim_F A = m^2$. Let Gr be the Grassmannian of m dimensional linear subspaces of A and let X be the subvariety (or rather subscheme) of those linear subspaces that are invariant under right multiplication in A. Thus, if E is a field, we have

$$X(E) = X_E(E) = \{L \subset A_E \mid \dim_E L = m,\ LA_E \subset L\},$$

where A_E is of course a notation for the central simple E algebra $A \otimes_F E$. We call X the Severi-Brauer variety of A. It is a smooth scheme over F. Again we get an equivalence between splitting of X and splitting of A:

$$X_E \simeq \mathbb{P}_E^{m-1} \Rightarrow X(E) \neq \emptyset \Rightarrow [E \text{ splits } A, \text{ i.e. } A_E \simeq M_m(E)] \Rightarrow X_E \simeq \mathbb{P}_E^{m-1}.$$

In particular, the function field F(X) of X is a splitting field (of transcendence degree m − 1 over F) of A because $X_{F(X)}$ has an obvious F(X) rational point associated with the generic point of X. In fact F(X) is a "generic splitting field" for A [2].

Another way to understand why the Severi-Brauer variety X and the central simple F algebra A are closely related is by descent theory [26]. Put $L = F^s$, the separable closure of F in $\overline{F}$. The algebra A is a form of $M_m(L)$, i.e. $A_L \simeq M_m(L)$, and the automorphism group of $M_m(L)$, as an L-algebra, is $PGL_m(L)$. Similarly X is a form of projective space $\mathbb{P}_L^{m-1}$, i.e. $X_L \simeq \mathbb{P}_L^{m-1}$, and the automorphism group of $\mathbb{P}_L^{m-1}$ as a scheme over L is again $PGL_m(L)$. As the automorphism groups are the same, the forms of $M_m(L)$ and the forms of $\mathbb{P}_L^{m-1}$ are classified by the same $H^1(\mathrm{Gal}(L:F), PGL_m(L))$. Indeed one checks that A and its Severi-Brauer scheme X correspond with the same cocycle. It is a good exercise to make this explicit for a cyclic algebra and to find the connection between F(X) and the reduced norm on A, as in the example at the beginning of this section. Compare also [16], [6].

§3. <u>THE IMPORTANCE OF CYCLIC EXTENSIONS.</u>

Cyclic extensions of the field F are of course closely related with cyclic algebras over F [1], [10]. Let p be a prime that divides n. Thus p is <u>not</u> the characteristic of F. First let E be a cyclic field extension of degree p of F. Choose a generator σ of G = Gal(E:F). As in many theories we have restriction and corestriction maps such that res·cor acts as the element $1 + \sigma + \ldots + \sigma^{p-1}$ of the group ring $\mathbb{Z}[G]$ and cor·res acts as (multiplication by) p = [E:F]. The restriction maps are the ordinary covariant maps that go for instance from Br(F) to Br(E), from $K_2(F)$ to $K_2(E)$, from $\ker(\alpha_{F,n})$ to $\ker(\alpha_{E,n})$, from $K_2(R)$ to

$K_2(R_E)$ when R is an F-algebra,... . The corestriction maps, also known as transfer or norm maps, go in the opposite direction. The relation cor.res = $[E:F]$ holds for any finite field extension. It is a special case of the projection formula which also tells $\mathrm{cor}(\{x,\mathrm{res}(a)\}) = \{\mathrm{cor}(x),a\}$ for $x \in E^*$, $a \in F^*$.

The main theorem involves groups that are annihilated by n. They may be studied one p-primary component at a time. That is why one may fix p. If E is an extension of F of degree prime to p, we see from cor.res = $[E:F]$ that res is injective on the p-primary component of relevant groups so that the extension $F \to E$ is "understood". This is why we may assume $\mu_p \subset F$. (Adjoining μ_p defines an extension of degree prime to p). Similarly we could reduce to the case where F is perfect. When studying the p-primary components one may as well assume that n is a power of p and it is also easy to reduce further to the case n = p [23]. We further assume $n = p$, $\mu_p \subset F$, and write α_F for $\alpha_{F,n}$.

Now consider a Galois extension of F. We view it as a limit of finite Galois extensions. Choosing a p-Sylow subgroup we decompose a finite Galois extension into two steps, one step with degree prime to p, the other with a p-group as Galois group. The p-group is solvable so that the second step may be broken up into a chain of cyclic extensions of degree p. That is how often we end up studying cyclic extensions of degree p.

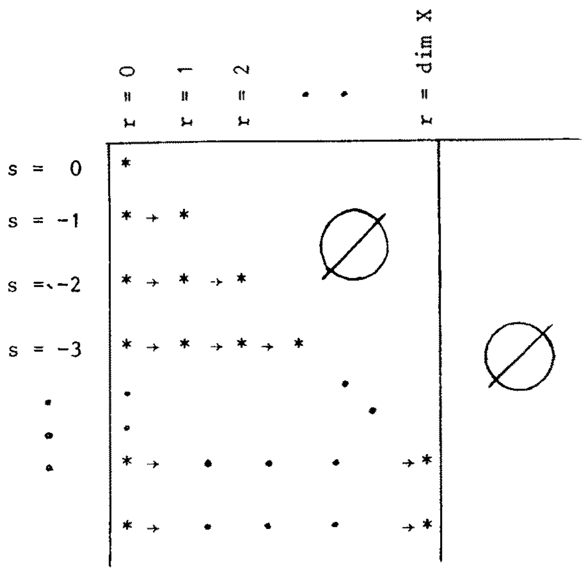
§4. COHOMOLOGY OF SHEAVES OF K-GROUPS

We now turn to the least elementary part, to wit K-cohomology. Quillen's constructions of algebraic K-groups are quite functorial and can therefore be sheafified. Let X be the Severi-Brauer variety of some non-split central simple F algebra A. (Note that F must be infinite, by Wedderburn, in order to have such A.) We get sheaves $\underline{\underline{K}}_n$ of abelian groups on X with the stalk at $x \in X$ of $\underline{\underline{K}}_n$ being K_n of the local ring $\mathcal{O}_{X,x}$ of x. In particular, the sheaf $\underline{\underline{K}}_0$ is the constant sheaf $\underline{\underline{\mathbb{Z}}}$, $\underline{\underline{K}}_1$ is the sheaf $\mathcal{O}_X^*$ of invertible local sections in the structure sheaf, and $\underline{\underline{K}}_2$ may be thought of as being obtained by sheafifying the presentation in Matsumoto's theorem: The sheaf map $\underline{\underline{K}}_1 \otimes_{\mathbb{Z}} \underline{\underline{K}}_1 \to \underline{\underline{K}}_2$ is surjective and the stalk at x of the kernel is generated by the $\{u\} \otimes \{1-u\}$ with $u, 1-u \in \mathcal{O}_{X,x}^*$. (We have avoided pathologies by making sure that the ground field is infinite).

We will be interested in $H^1(X,\underline{\underline{K}}_2)$. (Sheaf cohomology with respect to the Zariski topology.) In the original proof [15] of the Merkurjev-Suslin theorem one also needed to understand $H^0(X,\underline{\underline{K}}_2)$ and this required a long argument and very heavy machinery. But in Merkurjev's more elementary proof this part (and more) is avoided [14]. The cohomology of sheaves of K groups -also called K cohomology- is studied by means of the Brown-Gersten-Quillen spectral sequence

$$E_2^{rs} = H^r(X, \underline{\underline{K}}_{-s}) \Rightarrow K_{-r-s}(X).$$

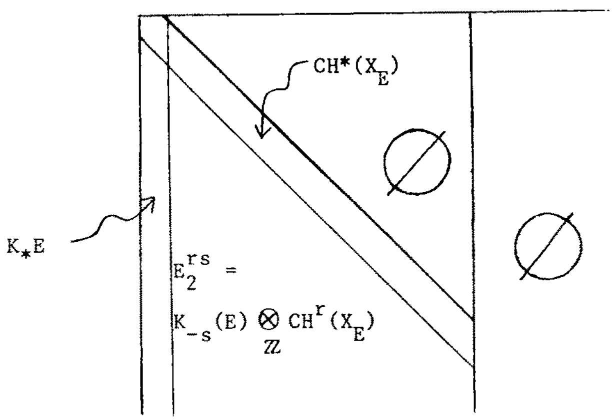
Its E_1 term looks like this.



The rows in the E_1 term are complexes, such as the following one in row $s = -2$ (symbolized in the picture by $* \to * \to *$).

$$0 \to K_2(F(X)) \to \coprod_{\substack{x \in X \\ \mathrm{codim}\{\overline{x}\}=1}} K_1(F(x)) \to \coprod_{\substack{x \in X \\ \mathrm{codim}\{\overline{x}\}=2}} K_0(F(x)) \to 0$$

Here $F(X)$ is the function field of X, $F(x)$ denotes the residue field of $\mathcal{O}_{X,x}$ and $\{\overline{x}\}$ the Zariski closure of the 1 point set $\{x\}$. To compute E_2^{*-2} one must take homology of the above complex. In particular $H^1(X, \underline{\underline{K}}_2)$ is the homology in the middle of the complex. (The maps in the complex can be made explicit and come from the localisation theorem.) The spectral sequence was studied by Quillen who also computed the abutment $K_*(X)$ in terms of the K-theory of the corresponding central simple algebra A [17]. Over a splitting field E the situation simplifies a lot. For $X_E \simeq \mathbb{P}_E^m$ the spectral sequence degenerates and the E_2 term becomes simply the tensor product of the Chow ring of X_E and the K theory of E [20], [8].



Here we see a special case of Bloch's formula which links K-theory with Chow groups:

$$H^r(X,\underline{\underline{K}}_r) \cong CH^r(X).$$

By Quillen [17] this formula is valid for X regular of finite type over a field. Recall that for $X_E = \mathbb{P}^m_E$ the Chow ring is very easy. It simply encodes the notion of the degree of a subvariety and Bezout's theorem on the degree of an intersection. Thus $CH^r(\mathbb{P}^m_E) = \mathbb{Z}$ for $0 \leqslant r \leqslant \dim \mathbb{P}^m_E = m$ and $CH^*(\mathbb{P}^m_E)$ is generated as a $\mathbb{Z}$ algebra by the class of the codimension 1 linear hyperplane. In any case the spectral sequence is sufficiently understood in the split case. Now let A be a cyclic algebra of dimension $n^2 = p^2$ (cf. §3), E a maximal subfield of A, hence a splitting field. One compares the spectral sequence for X with the one for X_E using restriction and corestriction maps and also basic properties of Grothendieck's Chern classes for K_0 "without denominators" [11], [9]. The result is that $H^1(X,\underline{\underline{K}}_2) \overset{res}{\to} H^1(X_E,\underline{\underline{K}}_2)$ is seen to be injective. That translates into a statement about the diagram

$$\begin{array}{ccccc} K_2(F(X)) & \to & \coprod_{x\ldots} K_1(F(x)) & \to & \coprod_{x\ldots} K_0(F(x)) \\ \downarrow res & & \downarrow res & & \downarrow res \\ K_2(E(X)) & \to & \coprod_{y\ldots} K_1(E(y)) & \to & \coprod_{y\ldots} K_0(E(y)) \end{array}$$

and that statement is what is needed in the next section.

§5. HILBERT 90 FOR K_2

Again let E be a cyclic extension of degree $n = p$ of F and let σ denote the generator of Gal(E:F). The important theorem concerning cyclic extensions and K_2, and a key step in the new proof of the main theorem, is:

THEOREM (HILBERT 90 FOR K_2) [15], [22].

The sequence $K_2E \overset{1-\sigma}{\to} K_2E \overset{cor}{\to} K_2F$ is exact.

In [22] it is shown that the theorem holds for any cyclic extension but we will only be concerned with the case $[E:F] = p$, $\mu_p \subset F$, cf. §3. Of course the theorem gets its name from the fact that one obtains Hilbert's Theorem 90 when replacing K_2 by K_1. Observe that the statement of the theorem involves nothing fancy like K cohomology. It would be very nice to have an elementary proof, as this would also yield an elementary proof of the main theorem. (The map cor can be made quite explicit via the projection formula [18].)

To prove the Hilbert 90 theorem one considers the homology H(L) of the complex $K_2(E_L) \xrightarrow{1-\sigma} K_2(E_L) \xrightarrow{cor} K_2L$ as a functor of L for field extensions $F \to L$. For instance, comparing H(F) with H(E) = 0 by restriction and corestriction one sees that H(F) is annihilated by p. If A is a non-split cyclic F algebra of dimension p^2 and X is the corresponding Severi-Brauer variety, then $H(F) \to H(F(X))$ is shown to be injective using the information obtained about the diagram at the end of §4. This is the place where K cohomology is used. Thus any non-trivial element of H(F) survives under $F \to F(X)$. It also survives under a finite field extension of degree prime to p (§3). Combining these two facts transfinitely often one sees that if F has the property $H(F) \neq 0$, one may preserve this property and enlarge F so that F has no finite extension of degree prime to p and all cyclic algebras of dimension p^2 over F split. (Recall F(X) splits A). But then F has become a very special kind of field (e.g. the norm cor: $K_1E \to K_1F$ must be surjective) and Hilbert 90 may be derived rather directly from Matsumoto's theorem in this special case. This contradicts $H(F) \neq 0$ and proves Hilbert 90 for K_2.

§6. THE PULL-BACK LEMMA

We keep p,σ,F,E as in §5. Using the equality $1 + \sigma + \dots + \sigma^{p-1} = (1-\sigma)^{p-1}$ in $\mathbb{Z}[\sigma]/p\mathbb{Z}[\sigma]$ and some computation one deduces from the Hilbert 90 theorem the following artificial lemma

PULL-BACK LEMMA. If α_F is injective then

$$\begin{array}{ccc} k_2F & \xrightarrow{\alpha_F} & {}_pBr(F) \\ \downarrow res & & \downarrow res \\ k_2E & \xrightarrow{\alpha_E} & {}_pBr(E) \end{array}$$

is a pull-back square.

Observe that the main theorem tells that α_F, α_E are actually isomorphisms so that the square is obviously a pull-back. But we don't yet have the main theorem. The lemma implies

PROPOSITION. If α_F is injective for all F, it is also surjective for all F. (All F of a fixed characteristic, distict from p.)

Proof. We claim that res: $coker(\alpha_F) \to coker(\alpha_L)$ is injective for any Galois extension $F \to L$. By section 3 the critical case is the cyclic case discussed in the pull-back lemma. Indeed in a pull-back square the vertical map between cokernels is injective. Now take for L the separable closure of F in $\overline{F}$. Then Br(L) = 0 and therefore $coker(\alpha_L) = 0$.

§7. SPECIALIZING FROM UNIVERSAL RELATIONS

We still have to get injectivity of α_F for all F. Here the idea is the same as in Merkurjev's original paper [13], where he proved Merkurjev's Theorem. He finds manageable universal reasons for the vanishing of an element $\sum_i \{\overline{a_i,b_i}\}$ in $k_2(F)$ (or $k_2(E)$). By clever manipulation of Matsumoto's theorem he shows that the only way $\Sigma \{\overline{a_i,b_i}\}$ can vanish in $k_2(F)$ is by a combination of the following two reasons.

1) The obvious reason: If a or b is a p-th power in F^*, then $\{\overline{a,b}\}$ vanishes in $k_2(F)$
2) The "Severi-Brauer reason": If X is the Severi-Brauer variety of a cyclic algebra $A_\omega(a,b)$ of dimension p^2 over F, then $\{\overline{a,b}\}$ vanishes in $k_2(F(X))$.

Here one may also describe the field F(X) explicitly, without reference to X. This description involves the norm cor: $K_1(F(\sqrt[p]{a})) \to K_1(F)$. To understand why $\{\overline{a,b}\}$ vanishes in $k_2(F(X))$, recall that F(X) splits $A_\omega(a,b)$ so that $\{\overline{a,b}\} \in \ker(\alpha_{F(X)})$. But it is well known that zero is the only element in the kernel of the Galois symbol that can be expressed as the coset of a single Steinberg symbol [16]. The problems lie with cosets of sums of several Steinberg symbols only. Thus $\{\overline{a,b}\}$ vanishes in $k_2(F(X))$. We may view the vanishing of $\{\overline{a,b}\}$ in $k_2(F(X))$ as a consequence, by specialization, of a universal case where a,b are transcendentals over the prime field. Thus we need specialization maps in K-theory that allow us to make substitution in transcendental variables. Indeed if x is a smooth point on a variety (scheme) X over a field L one has (unfortunately not canonical) specialization homomorphisms $K_i(L(X)) \to K_i(L(x))$ so that composition with $K_i(\mathcal{O}_{X,x}) \xrightarrow{res} K_i(L(X))$ is the ordinary res: $K_i(\mathcal{O}_{X,x}) \to K_i(L(x))$ coming from the residue homomorphism $\mathcal{O}_{X,x} \to L(x)$ [8, Def.8.2]. For k_2 one may give a direct construction of such specialization maps, and this construction works for any field with valuation [24]. It is in the sense of specialization that Merkurjev finds universal reasons for vanishing. That is, he finds a countable family of universal cases, each involving a finite product of Severi-Brauer varieties, such that if $\Sigma \{\overline{a_i,b_i}\}$ vanishes in $k_2(F)$, it is obtained by specialization from one of the universal cases. (The number of Severi-Brauer varieties of which one has to take a product is simply the number of times a "Severi-Brauer reason" is applied.)

§8. THE KERNEL OF $k_2F \to k_2E$

We take F as usual, $E = F(\sqrt[p]{a})$ a cyclic extension ($a \in F^*$, a not a p-th power). If we knew that α_F is injective, the pull-back lemma (§6) would tell that $\ker(k_2F \to k_2E)$ corresponds in Br(F) with the similarity classes of central simple F algebras containing E as a maximal subfield, hence that $\ker(k_2F \to k_2E)$ consists of the $\{\overline{a,b}\}$ with $b \in F^*$.

That last statement Merkurjev proves by reduction to a universal case, using his universal reasons for vanishing. The case one reduces to is the case where F is a Galois extension, of degree a power of p, of a purely transcendental extension of a global field. This situation is easily handled by a combination of the pull-back lemma and the earlier work of Tate and Bloch. Tate proved [23] that the Galois symbol is an isomorphism for global fields and Bloch [5], [7] showed that it then also is an isomorphism for purely transcendental extensions of global fields.

§9. INJECTIVITY OF α_F

The proof of the main theorem is finished like this. The previous section tells us for a cyclic extension E of F of degree p that $\ker(k_2F \to k_2E)$ consists of cosets of single Steinberg symbols. Therefore (compare §7) this kernel has zero intersection with $\ker \alpha_F$. In other words, res: $\ker(\alpha_F) \to \ker(\alpha_E)$ is injective for such cyclic extensions. But then res: $\ker(\alpha_F) \to \ker(\alpha_L)$ is injective for all Galois extensions $F \to L$, by the reductions of §3. We take for L the separable closure of F in $\overline{F}$ and observe that L^* consists of p-th powers so that $k_2L = 0$ and $\ker(\alpha_L) = 0$. Done.

REMARK. Merkurjev's new proof, as described here, is much shorter then the proof in [15]. It also establishes different logical connections so that there are now several ways to reach the main theorem. For instance, one may now reprove Tate's Theorem (saying that the Galois symbol is an isomorphism for a global field) without using class field theory. (One would need higher Chern classes for K cohomology, etcetera, and a lot of work, so that I cannot recommend this approach.)

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Added in print:

Correction to ref. [14]

[14] A.S. Merkurjev, K_2 of fields and the Brauer Group, to appear in Contemporary Mathematics, American Math. Soc.