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A Group Structure on Certain Orbit Sets of Unimodular Rows

WILBERD VAN DER KALLEN

Math. Instituut, University of Utrecht, Postbus 80.010, 3508 TA Utrecht, The Netherlands

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1. INTRODUCTION

Let R be a commutative noetherian d-dimensional ring. Recall that for $n \geqslant d+2$ the group $E_n(R)$ (the subgroup of $GL_n(R)$ generated by elementary matrices) acts transitively on $Um_n(R)$, the set of unimodular rows of length n over R. If $d \geqslant 2$, we will describe an abelian group structure on $Um_{d+1}(R)/E_{d+1}(R)$. This group structure will be closely related with the higher Mennicke symbols of Suslin. In fact this article is mainly an elaboration of a theme in Suslin [14] (in particular [14, Sect. 1]). Recall that for $d=1$, by the Bass–Kubota theorem there is a bijection $MS_2(R) \leftrightarrow Um_2(R)/SL_2(R) \cap E(R)$, where $MS_2(R)$ denotes the target group of the universal Mennicke symbol, as in Suslin [13, Sect. 5]. We will see that more generally $MS_{d+1}(R) \leftrightarrow Um_{d+1}(R)/SL_{d+1}(R) \cap E(R)$ if d is odd. Now let $d=2$. Then, by a theorem of Vaserstein, $Um_3(R)/E_3(R)$ is in bijective correspondence with a certain Witt group [17, Cor. 7.4]. In particular $Um_3(R)/E_3(R)$ gets the structure of an abelian group. We will derive from this (inductively) the structure of an abelian group on $Um_{d+1}(R)/E_{d+1}(R)$ for $d \geqslant 3$. (It would be desirable to have an interpretation of these abelian groups in terms of Witt groups or of similar Grothendieck groups of categories.)

As an intriguing by-product we get an abelian group structure on the set of isomorphism classes of oriented stably free rank d projective modules. (If d is odd one does not need the orientations. See 4.8.)

We will borrow heavily from the work of Suslin and Vaserstein. For the convenience of the reader we have included a proof of Vaserstein's prestabilization theorem for K_1, making no restriction on the presence of zero-divisors. (In Vaserstein's original proof such restrictions were made, but he has long since been able to remove them. Because of the crucial role his theorem plays in connecting higher Mennicke symbols with ordinary K-

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groups and with the group constructed in this article, an updated exposition seemed desirable.)

2. Vaserstein's Pre-stabilization Theorem for K_1

(2.1). Let A be an associative ring with 1 and let I be a two-sided ideal in A. For $n \geqslant 2$ we define $E_n^1(A, I)$ to be the subgroup of $GL_n(A)$ generated by the $e_{1i}(a)$ with $a \in A$, $2 \leqslant i \leqslant n$, and the $e_{i1}(x)$, $x \in I$, $2 \leqslant i \leqslant n$. Recall that $E_n(A, I)$ denotes the smallest normal subgroup of $E_n(A)$ containing the elements $e_{21}(x)$ with $x \in I$. As is pointed out in [17, Sect. 2], $E_n(A, I)$ is generated by elements $e_{ij}(a)\, e_{ji}(x)\, e_{ij}(-a)$ with $a \in A$, $x \in I$, $i \neq j$, provided that $n \geqslant 3$. (The proof (see [21, Lemma 8]), exploits the Steinberg relations, as in the next proof.) Recall that $GL_n(A, I) = \ker(GL_n(A) \to GL_n(A/I))$.

(2.2) Lemma. *For $n \geqslant 3$ the following sequence is exact.*

$$1 \to E_n(A, I) \to E_n^1(A, I) \to E_n^1(A/I, 0) \to 1.$$

Thus $E_n(A, I)$ equals $E_n^1(A, I) \cap GL_n(A, I)$.

Proof. It suffices to show that $E_n^1(A, I)$ contains the sets $S_{ij} = \{e_{ij}(a)\, e_{ji}(x)\, e_{ij}(-a) : a \in A,\ x \in I\}$ for all roots ij (cf. 2.1). In the following computation we show that if $E_n^1(A, I)$ contains S_{ik} and S_{jk} it contains S_{ij}. We write $*$ for some elements of I and of $E_n^1(A, I)$ and we use the standard identities $[gh, k] = {}^g[h, k][g, k]$, $[g, hk] = [g, h]^h\, [g, k]$, where ${}^x y$ denotes xyx^{-1} and $[x, y]$ denotes ${}^x y \cdot y^{-1}$, as usual. Let $x \in I$, $a \in A$. We get (compare also [6, 3.5])

$$\begin{aligned}
{}^{e_{ij}(a)}e_{ji}(x) &= {}^{e_{ij}(a)}[e_{jk}(1), e_{ki}(*)] \\
&= [{}^{e_{ij}(a)}e_{jk}(1), {}^{e_{ij}(a)}e_{ki}(*)] \\
&= [e_{ik}(a)\, e_{jk}(1), e_{ki}(*)\, e_{kj}(*)] \\
&= {}^{e_{ik}(a)}[e_{jk}(1), e_{ki}(*)\, e_{kj}(*)][e_{ik}(a), e_{ki}(*)\, e_{kj}(*)] \\
&= {}^{e_{ik}(a)}e_{ji}(*)\ {}^{e_{ik}(a) e_{ki}(*)}[e_{jk}(1), e_{kj}(*)][e_{ik}(a), e_{ki}(*)]\, e_{kj}(*)\, e_{ij}(*) \\
&= (*)\ {}^{e_{ik}(a)}(e_{ki}(*)\, e_{ji}(*)[e_{jk}(1), e_{kj}(*)])[e_{ik}(a), e_{ki}(*)](*) \\
&= (*)[e_{ik}(a), e_{ki}(*)](*)[e_{jk}(1), e_{kj}(*)](*)[e_{ik}(a), e_{ki}(*)](*),
\end{aligned}$$

which lies in the group generated by $E_n^1(A, I)$, S_{ik}, S_{jk}. Similarly, if $E_n^1(A, I)$ contains S_{ki} and S_{kj} it contains S_{ji}. Therefore, as it contains S_{12}, S_{13}, it contains S_{23}, S_{32}, S_{21}, S_{31}. And so on.

Remark. The same type of argument may be used in the Steinberg group $St_n(A)$.

(2.3) *Conventions.* In the rest of this article we make the following assumptions (cf. [2, Chap. IV, V]):

The ring A is finitely generated as a module over a central subring R. There is an integer d so that the maximal spectrum of R is the union of $V(I)=\{\mathfrak{m}$: the image of I in $A/\mathfrak{m}A$ is contained in the Jacobson radical of $A/\mathfrak{m}A\}$ and finitely many subsets V_i, where each V_i, when endowed with the (topology induced from the) Zariski topology is a noetherian space of dimension $\leqslant d$. (The reader may feel more comfortable with the special case where $A=I=R$ is a commutative noetherian ring of dimension d). As a subspace of a noetherian space X never has larger dimension than X, we further assume that the V_i are disjoint from each other and from $V(I)$.

(2.4) The hypotheses in (2.3) are designed to make Bass' Stable Range Theorem apply to the ideal I, i.e., to make that *st.r.* $(\mathrm{I})\leqslant d+1$ in the terminology of [19]. In fact we have the following result, which may appear stronger, but is actually equivalent to *st.r.*$(I)\leqslant d+1$.

PROPOSITION (cf. [20, Thm. 2.3(e), Thm. 2.5]). *Let* $(a_0,\dots,a_m)\in Um_{m+1}(A)$, $m\geqslant d+1$, $a_0-1\in I$. *Let* S *be a subset of size* $d+1$ *of* $\{0,\dots,m-1\}$. *Then there are* $t_i\in I$ *with* $t_i=0$ *for* $i\notin S$ *such that* $(a_0+a_mt_0,\dots,a_{m-1}+a_mt_{m-1})\in Um_m(A)$.

Proof. We argue by induction on d as in the proof of [20, Thm. 2.5] (compare also [2, proof of V Thm. 3.5]). First let us check the result when R is a field. We may compute modulo the Jacobson radical, so we may as well assume that A is simple. But then the ideal I equals 0 or A. If $I=0$, then $a_0=1$. If $I=A$, use [19, Thm. 1, Cor. to Thm. 3]. More generally the case $d=0$ is easy, as one has to deal with only finitely many points outside $V(I)$. (See also what follows.) Now let $d\geqslant 1$. Choose $\mathfrak{m}_1,\dots,\mathfrak{m}_g$ so that each irreducible component of each V_i contains at least one $\mathfrak{m}_j$. Say $s\in S$, $s\neq 0$. We now wish to choose $r_i\in A$ so that $r_m\in I$ and so that $\tilde{a}_s=a_s+\sum_{i\neq s}a_ir_i$ is a unit in $A\otimes R/\mathfrak{m}_j$ for $j=1,\dots,g$. To see that this is possible, consider the image $\bar{I}$ of I in $\bar{A}=A\otimes R/\mathfrak{m}_1\cap\cdots\cap\mathfrak{m}_g=\prod_j A\otimes R/\mathfrak{m}_j$ and observe that we can use the argument above in each factor $A\otimes R/\mathfrak{m}_j$, as $R/\mathfrak{m}_j$ is a field. Now compute as in [20, proof Theorem 3.5]: One finds q in $R\cap\tilde{a}_sA$, q not in any of the $\mathfrak{m}_j$, applies the inductive assumption to the image of I in A/qA, with s deleted from S, and so on.

(2.5) Contrary to [20] we will further restrict ourselves to the case $d\geqslant 1$.

(2.6) *Notations.* $M_n(A)$ is the ring of n by n matrices over A. $M_n(I)=\ker(M_n(A)\to M_n(A/I))$, $GL_n(A,I)=\ker(GL_n(A)\to GL_n(A/I))$. If

$X = 1_n + Y \cdot \mathrm{diag}(q, 1,\ldots, 1)$ is invertible, write $\tilde{X}_{q,Y}$, or simply $\tilde{X}_q$, for the invertible matrix $1_n + \mathrm{diag}(q, 1,\ldots, 1) \cdot Y$(cf. [4, 8.4; 20, Sect. 1]). Let $\tilde{E}_n^1(A, I)$ be the subgroup of $GL_n(A)$ generated by $E_n^1(A, I)$ and the $\tilde{X}_{q,Y}$ with $X = 1_n + Y \cdot \mathrm{diag}(q, 1,\ldots, 1)$, $X \in E_n^1(A, I)$, $Y \in M_n(I)$, $q \in A$. We intend to prove the following version of Vaserstein's pre-stabilization theorem.

(2.7) THEOREM. *Let $d \geqslant 1$ (cf. (2.3)). Then*

$$E_{d+2}(A, I) \cap GL_{d+1}(A) = \tilde{E}_{d+1}^1(A, I) \cap GL_{d+1}(A, I).$$

(2.8) COROLLARY (Vaserstein's Pre-stabilization Theorem, cf. [20, (3.4)]). *The kernel of the surjective homomorphism $GL_{d+1}(A, I) \to K_1(A, I)$ is the smallest subgroup H of $GL_{d+1}(A)$ normalized by $E_{d+1}(A)$ and containing all matrices of the form $(1 + DY)(1 + YD)^{-1}$ with $D = \mathrm{diag}(q, 1,\ldots, 1)$ for some $q \in A$, $Y \in M_{d+1}(I)$, $1 + DY \in GL_{d+1}(A, I)$.*

Proof of Corollary. $E_{d+1}(A, I)$ is contained in H and H is contained in the kernel, by [20, Sect. 1]. We have to show that the kernel is not larger than H. By stability for K_1 ([20, Theorem 3.2]; see also [16, Thm. 2.2]), the kernel equals $E_{d+2}(A, I) \cap GL_{d+1}(A)$ and the rest is easy (use (2.2)).

(2.9) *Remarks.* (a) For $n \geqslant 3$ one has $[E_n(A), GL_n(A, I)] = E_n(A, I)$ (see [21, Cor. 14]). Hence for $d \geqslant 2$ the group H of the corollary is actually generated by $E_{d+1}(A, I)$ and the matrices $(1 + DY)(1 + YD)^{-1}$ of the corollary.

(b) For $d = 1$ the pre-stabilization theorem implies the Bass–Kubota theorem (cf. [17, Sect. 16, Remark (b)]), as follows. As in [13, Sect. 5], let $MS_n(A, I)$ denote the target group of the universal Mennicke symbol on $Um_n(A, I)$ and let $ms(v)$ denote the class of v.

THEOREM (Kubota). *Let $d = 1$ and $A = R$ (cf. (2.3)). The map ms: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ms(a, b)$ defines a homomorphism $GL_2(A, I) \to MS_2(A, I)$.*

Proof. The proof in [4] is given for the case that A is a 1-dimensional domain. As is well known, one can easily deal with zero divisors. Before sketching the argument we give some definitions.

(2.10) DEFINITIONS. If X is an irreducible component of dimension d of one of the V_i (see (2.3)), we say that the prime ideal $\mathfrak{p} = \bigcap_{\mathfrak{m} \in X} \mathfrak{m}$ is a *critical* prime ideal. There are finitely many critical primes. If $\mathfrak{p}$ is critical then $R/\mathfrak{p}$ is infinite, because $d \geqslant 1$. Recall that this fact is useful for achieving "multiple" goals (compare [6, Sect. 2, 6; 7]). In this article we say that an element a of A is *in general position* if for each critical $\mathfrak{p}$, $a \otimes 1$ is a unit in $A(\mathfrak{p}) = A \otimes_R k(\mathfrak{p})$, where $k(\mathfrak{p})$ is the field of fractions of $R/\mathfrak{p}$. If a is in general position, there is an $x \in A$ such that $xa \in R$, $xa \notin \mathfrak{p}$ for each critical $\mathfrak{p}$. Observe that $\operatorname{spec}(R/xaR)$ is the union of $V(I + xaA/xaA)$ and finitely many subspaces of dimension at most $d-1$.

(2.11) Let us return to the proof of Kubota's theorem. Thus $d = 1$. Let

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \alpha' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

be elements of $GL_2(A, I)$. From [4, Sect. 6] one sees

(i) If $g \in E_2(A)$ then $ms(g\alpha g^{-1}) = ms(\alpha)$,

(ii) If $h \in E_2(A, I)$ then $ms(\alpha h) = ms(h\alpha) = ms(\alpha)$,

(iii) If $d \equiv 1 \mod (a'-1)$ and $dA + a'A = A$, then $ms(\alpha'\alpha) = ms(\alpha')\, ms(\alpha)$.

To show that $ms(\alpha'\alpha) = ms(\alpha')\, ms(\alpha)$ in the general case, we adapt page 105 of [4] as follows. First multiply α' by an element of $E_2(I)$ so as to reduce to the case that a', $a'-1$ are in general position. Then multiply α (from the right) by an element of $E_2(A, I)$ so as to achieve $d \equiv 1 \mod (a'-1)$. (By Prop. 2.4 we may use [20, Thm. 2.3(d)] to see that $SL_2(A, I) = SL_2(A, (a'-1)A)\, E_2(A, I)$.) Finally multiply α from the right by an element $e_{12}(x)$, $x \in (a'-1)A \subseteq I$ to reduce to case (iii).

(2.12) Theorem (Bass–Kubota). *Let $d = 1$, $A = R$. Then ms induces an isomorphism $SK_1(A, I) \to MS_2(A, I)$.*

Proof. By 2.11(i) the kernel of $ms\colon GL_2(A, I) \to MS_2(A, I)$ is normalized by $E_2(A)$. If $Y \in M_2(I)$, $q \in A$, $D = \operatorname{diag}(q, 1)$ with $1 + DY$ invertible, then clearly $ms(1 + DY) = ms(1 + YD)$. By Corollary 2.8 we therefore get a homomorphism $K_1(A, I) \to MS_2(A, I)$, induced by ms. Now compare with the usual homomorphism $MS_2(A, I) \to SK_1(A, I)$, as in [4, Sect. 5]: cf. [9, Sect. 13].

(2.13) To prove Theorem 2.7 we introduce a normal form. Put

$$P = \begin{pmatrix} 0 & 1_d \\ -1 & 0 \end{pmatrix}.$$

Let C (cf. [6, 7, 8]) denote the set of elements of $GL_{d+2}(A)$ that can be written as $\left(\begin{smallmatrix} M & 0 \\ v & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & w \\ 0 & N \end{smallmatrix}\right)$ with $M \in E^1_{d+1}(A, I) \cap GL_{d+1}(A, I)$, v a row with entries in A, w a row with entries in I, $P^{-1}NP \in \tilde{E}^1_{d+1}(A, I)$. Note that modulo I every element of C has the form

$$\begin{pmatrix} 1_{d+1} & 0 \\ * & 1 \end{pmatrix}.$$

(2.14) LEMMA (cf. [20, Sect. 1]). *Let* $X = 1_{d+1} + YD \in GL_{d+1}(A)$, *with* $D = \operatorname{diag}(q, 1, ..., 1)$, $q \in A$, *as usual. Let b denote the first column of Y. Then*

$$\begin{pmatrix} 1_{d+1} & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_d & 0 \\ q & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_d & 0 \\ q & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\tilde{X}_q)^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1_{d+1} \end{pmatrix}$$

In particular, if $Y \in M_{d+1}(I)$, $X \in E^1_{d+1}(A, I)$, *then* $\tilde{X}_q \in \tilde{E}^1_{d+1}(A, I)$ *and the matrix is in* C.

Proof. Say $Y = \left(\begin{smallmatrix} u & v \\ w & M \end{smallmatrix}\right)$ with $M \in M_d(A)$. Then

$$\begin{pmatrix} 1_{d+1} & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_d & 0 \\ q & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v & -u \\ 0 & 1+M & -w \\ 0 & -qv & 1+qu \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_d & 0 \\ q & 0 & 1 \end{pmatrix}.$$

(2.15) PROPOSITION. *Let* $d \geqslant 1$ *and* $E = e_1 e_2 \cdots e_t$, *where*

$$e_i \in \{e_{d+2,p}(a) : a \in A,\ 1 \leqslant p \leqslant d+1\}$$
$$\cup \{e_{p,d+2}(x) : x \in I,\ 1 \leqslant p \leqslant d+2\}.$$

Choose a row u, with entries in A, so that the lower left hand corner of

$$\begin{pmatrix} 1_{d+1} & 0 \\ u & 1 \end{pmatrix} e_1 \cdots e_i$$

is in general position (*see* 2.10) *for* $i = 0, 1, ..., t$. *Then there is a column v so that*

$$\begin{pmatrix} 1_{d+1} & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{d+1} & 0 \\ u & 1 \end{pmatrix} E \in C.$$

Proof. It clearly suffices to show: If $Y \in C$, e is one of the e_i, and the lower left hand corners of Y, Ye are both in general position, then there is a column v such that

$$\begin{pmatrix} 1_{d+1} & v \\ 0 & 1 \end{pmatrix} Ye \in C.$$

Note further that we may assume $e = e_{d+2,1}(*)$, the other cases being easy. Write Y as

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_d & 0 \\ q & r & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & N \end{pmatrix}$$

with $M \in E^1_{d+1}(A, I) \cap GL_{d+1}(A, I)$, $w \equiv 0 \bmod I$, $P^{-1}NP \in \tilde{E}^1_{d+1}(A, I)$. As M normalizes the group of the

$$\begin{pmatrix} 1_{d+1} & * \\ 0 & 1 \end{pmatrix},$$

and C is invariant under left multiplication by $M^{\pm 1}$, we may replace M by 1_{d+1}. As we may also absorb

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{pmatrix}$$

into $\left(\begin{smallmatrix} 1 & w \\ 0 & N \end{smallmatrix}\right)$ we may take r to be zero, so that

$$Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & N \end{pmatrix}.$$

Write the first column of Ye as

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ z \end{pmatrix},$$

where a_0, a_2, z are scalars and a_1 is a column of length $d-1$. Choose $s \in Az \cap R$ such that s is not in any of the critical primes. Choose r_0, r_1, both congruent to zero mod I, so that

$$\begin{pmatrix} a_0 + r_0 a_2 \\ a_1 + r_1 a_2 \\ z \end{pmatrix}$$

is unimodular. (Modulo s one may take a lower value for d and then apply Proposition 2.4 to the transpose of the column; cf. [20, Thm. 2.6].) Say

$$(b_0, b_1, y)\begin{pmatrix} a_0 + r_0 a_2 \\ a_1 + r_1 a_2 \\ z \end{pmatrix} = 1.$$

Put $\lambda = (a_0 - 1 + r_0 a_2 - a_2) y$ and put

$$v = \begin{pmatrix} -r_0 \lambda \\ -r_1 \lambda \\ \lambda \end{pmatrix}.$$

Note that the entries of v are in I. We have to show that $\left(\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right) Ye \in C$. As

$$\begin{pmatrix} 1 - r_0 \lambda q & 0 & -r_0 \\ -r_1 \lambda q & 1_{d-1} & 0 \\ \lambda q & 0 & 1 \end{pmatrix} \in E^1_{d+1}(A, I),$$

it follows from Lemma 2.14 that $\left(\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right) Y \in C$. Say

$$\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} Y = \begin{pmatrix} M_1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & t \\ 0 & 1_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & N_1 \end{pmatrix}$$

with $M_1 \in E^1_{d+1}(A, I) \cap GL_{d+1}(A, I)$, etc. It suffices to show now that

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & N_1 \end{pmatrix} e \in C,$$

or that

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ g & 1_d & 0 \\ h & 0 & 1 \end{pmatrix} \in C,$$

where

$$\begin{pmatrix} 1 & 0 & 0 \\ g & 1 & 0 \\ h & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & N_1 \end{pmatrix} e \begin{pmatrix} 1 & 0 \\ 0 & N_1^{-1} \end{pmatrix}.$$

Put

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1_{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -a_1 - r_1 a_2 & 1_{d-1} & 0 \\ 1 - a_0 - r_0 a_2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1_{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{d-1} & 0 \\ \mu b_0 & \mu b_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & r_0 \\ 0 & 1_{d-1} & r_1 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\mu = a_0 - 1 + r_0 a_2 - a_2$. Note that $B \in E^1_{d+1}(A, I) \cap GL_{d+1}(A, I)$ and that

$$\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} Ye = \begin{pmatrix} 1 & \\ 0 & * \\ 0 & \\ z & \end{pmatrix}.$$

Therefore the first column of the element

$$\begin{pmatrix} 1 & 0 \\ -g & 1_d \end{pmatrix} M_1^{-1} B^{-1}$$

of $E^1_{d+1}(A,I)\cap GL_{d+1}(A,I)$ is equal to the top part of the first column of

$$\begin{pmatrix} 1 & 0 & 0 \\ -g & 1_d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} M_1^{-1} & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} Ye$$

$$= \begin{pmatrix} 1 & 0 & t \\ 0 & 1_d & -gt \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ 0 & * \\ h & \end{pmatrix}.$$

Applying Lemma 2.14 again we see that

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1_d & -gt \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_d & 0 \\ h & 0 & 1 \end{pmatrix} \in C$$

and that does it, as

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ g & 1 & 0 \\ h & 0 & 1 \end{pmatrix}$$

is just

$$\begin{pmatrix} 1 & 0 & 0 \\ g & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

times this element.

(2.16) Next let us prove Theorem 2.7. First consider a generator of type $\tilde{X}_{q,Y}$ of $\tilde{E}^1_{d+2}(A,I)$. By Lemma 2.2 the corresponding X lies in $E_{d+2}(A,I)$ and we see from Lemma 2.14, viewed as giving a relation in $E_{d+2}(A)/E_{d+2}(A,I)$, that $\tilde{X}_{q,Y}$ lies in $E_{d+2}(A,I)$ too (cf. [20, Sect. 1]). Thus the right hand side in Theorem 2.7 is contained in the group $E^1_{d+2}(A,I)\cap GL_{d+2}(A,I)$; hence by 2.2 it is contained in the left hand side. To derive the reverse inclusion, consider an element E' of $E_{d+2}(A,I)\cap GL_{d+1}(A)$.

Put

$$E = \begin{pmatrix} 0 & 1 \\ P & 0 \end{pmatrix} E' \begin{pmatrix} 0 & 1 \\ P & 0 \end{pmatrix}^{-1},$$

say $E=\begin{pmatrix}1&0\\0&T\end{pmatrix}$, $E\in E_{d+2}(A,I)$. By Lemma 2.2, with the indices 1 and $d+2$ interchanged, we may write $E=e_1\cdots e_t$ as in Proposition 2.15. Thus we have $\begin{pmatrix}1&v\\0&1\end{pmatrix}\begin{pmatrix}1&0\\u&1\end{pmatrix}\ E=\begin{pmatrix}M&0\\ *&1\end{pmatrix}\begin{pmatrix}1&*\\0&N\end{pmatrix}$ with $M\in E^1_{d+1}(A,I)\cap GL_{d+1}(A,I)$, $P^{-1}NP\in \tilde{E}^1_{d+1}(A,I)$, u a suitable row, v a suitable column. Write $\begin{pmatrix}1&0\\u&1\end{pmatrix}$ as

$$\begin{pmatrix}1&0&0\\0&1_d&0\\q&r&1\end{pmatrix}.$$

Then by Lemma 2.14,

$$\begin{pmatrix}1&v\\0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1_d&0\\q&0&1\end{pmatrix}$$

$$=\begin{pmatrix}M&0\\0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\q&0&1\end{pmatrix}\begin{pmatrix}1&0\\0&P\end{pmatrix}\begin{pmatrix}1&0\\0&(\tilde{M}_q)^{-1}\end{pmatrix}\begin{pmatrix}1&0\\0&P^{-1}\end{pmatrix}\begin{pmatrix}1&*\\0&1_{d+1}\end{pmatrix}.$$

So

$$\begin{pmatrix}1&0\\0&T\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&-r&1\end{pmatrix}\begin{pmatrix}1&*\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&P\end{pmatrix}\begin{pmatrix}1&0\\0&\tilde{M}_q\end{pmatrix}\begin{pmatrix}1&0\\0&P^{-1}\end{pmatrix}$$

$$\times\begin{pmatrix}1&0&0\\0&1&0\\-q&0&1\end{pmatrix}\begin{pmatrix}M^{-1}&0\\0&1\end{pmatrix}\begin{pmatrix}M&0\\ *&1\end{pmatrix}\begin{pmatrix}1&*\\0&N\end{pmatrix}.$$

Cancel $M^{-1}M$, compare first columns and then first rows. The result follows.

(2.17) THEOREM. *Let $d\geqslant 1$, $A=R$. Then $E_{d+2}(A,I)\cap GL_{d+1}(A,I)$ is generated by $E^1_{d+1}(A,I)\cap GL_{d+1}(A,I)$ and the $1_{d+1}+DY$ with $D=diag(q,1,\dots,1)$, $q\in I$, $1+YD\in E^1_{d+1}(A,I)$, $1+DY\in GL_{d+1}(A,I)$.*

(2.18) *Remarks.* $E^1_2(A,I)\cap GL_2(A,I)$ is generated by the $e_{12}(a)\,e_{21}(x)\,e_{12}(-a)$ with $x\in I$, $a\in A$, and the $e_{12}(x)$ with $x\in I$. For $n\geqslant 3$, $E^1_n(A,I)\cap GL_n(A,I)$ is just $E_n(A,I)$, by Lemma 2.2.

(2.19) Theorem 2.17 is proved in the same fashion as Theorem 2.7. Now one takes $C=\{\begin{pmatrix}M&0\\v&1\end{pmatrix}\begin{pmatrix}1&w\\0&N\end{pmatrix}: M\in E^1_n(A,I),\ v\equiv 0 \bmod I,\ P^{-1}NP\in$ the group generated by the elements mentioned in 2.17}. In the analogue of Proposition 2.15 one takes the elementary matrices e_i to be in $E^1_{d+2}(A, I)$. We may leave the details to the reader.

(2.20) *Remark.* It follows from the proof of Theorem 2.17 that one may replace the condition $A = R$ by the weaker condition that I contains an element in general position. Examples of this situation are

(1) R is a d-dimensional noetherian domain and $R \cap I \neq 0$.

(2) The image of I in $\bar{A} = A/(I \cap R)A$ is contained in the Jacobson radical of $\bar{A}$.

3. The Groups $MSE_{d+1}(R, I)$

In this section we prove the main result (Theorem 3.6). We prove the existence of a group structure on certain orbit sets, first in the absolute case, then relative to an ideal. It turns out that the relative case follows from the absolute case because the orbit sets satisfy an excision property (Theorem 3.21). Having constructed a multiplicative structure we explore its calculus and show that many known formulas have their counterpart in our setting (Theorems 3.6, 3.16, 3.25).

(3.1) From now on we will only consider commutative rings. In 2.3 we therefore take $A = R$. Note that $V(I)$ now equals $\{\mathfrak{m}$: $\mathfrak{m}$ is maximal ideal of R containing $I\}$.

(3.2) *Notations.* If J is an ideal in the (commutative) ring B and $n \geqslant 2$, write $Um_n(B, J)$ or $Um_n(J)$ for the set of J-unimodular rows of length n. Recall that $Um_n(J)$ does not depend, up to natural bijections, on the ambient ring B [19, Lemma 1]. If $n \geqslant 3$, write $MSE_n(B, J)$ for the orbit set $Um_n(J)/E_n(B, J)$. Write $MSE_2(B, J)$ for $Um_2(J)/T_2(B, J)$, where $T_2(B, J)$ is the subgroup of $SL_2(B)$ generated by matrices

$$\begin{pmatrix} 1-abt & -b^2t \\ a^2t & 1+abt \end{pmatrix}$$

with $a,\ b \in B,\ t \in J$. One checks that $T_2(B, J)$ is a normal subgroup of $GL_2(B)$, containing $E_2(B, J)$. (In fact the generating set is invariant under conjugation by elements of $GL_2(B)$.) Note further that $T_2(B, J) \subset E_3(B, J)$, because

$$\begin{pmatrix} 1-abt & -bt \\ a^2bt & 1+abt \end{pmatrix} = e_{21}(a)^{-1}\, e_{12}(-bt)\, e_{21}(a) \in E_2(B, J).$$

(Use Mennicke relations or [20, Lemma 1.1] or Lemma 2.14.) If $v \in Um_n(B, J)$, write $mse(v)$ for its orbit under $E_n(B, J)$ (resp. $T_2(B, J)$ if $n = 2$). If $g \in GL_n(B)$ has first row $v \in Um_n(J)$, write $mse(g)$ for $mse(v)$. The usual argument (cf. [4, Sect. 5]) shows that $g \mapsto mse(g)$ induces a bijection $SL_2(B, J)/T_2(B, J) \to MSE_2(B, J)$. By way of this bijection we may give $MSE_2(B, J)$ the structure of a group. We wish to generalize this. Some hypotheses on the ring are needed for the generalization. The case of $MSE_3(B)$ has been treated by Vaserstein [17, Sects. 5, 7]. (If $n \geqslant 3$, $MSE_n(B)$ stands for $MSE_n(B, B) = Um_n(B)/E_n(B)$, of course.)

(3.3) Recall that $A = R$ (see 3.1). As the setting of 2.3 is somewhat more complicated than the one in [13], we need a replacement for conditions of the type $ht(a_1 R + \cdots a_d R) \geqslant d$.

DEFINITION (cf. 2.10). We say that $(a_1, \ldots, a_r)$ is in *general position* if $V(a_1 R + \cdots + a_r R)$ is the union of $V(I + a_1 R + \cdots + a_r R)$ and finitely many subsets of dimension $\leqslant d - r$. Note that if a is in general position in the sense of 2.10, (a) is in general position in the sense of the present definition.

(3.4) LEMMA (cf. [14, Lemma 1.2; 4, Lemma 2.4]). *Let $v, w \in Um_{d+1}(R, I)$. There exist $\alpha, \beta \in E_{d+1}(R, I)$ such that $v \cdot \alpha = (a_0, a_1, \ldots, a_d)$, $w \cdot \beta = (b_0, a_1, \ldots, a_d)$ with $(a_1, \ldots, a_d)$ in general position.*

Proof. By induction on d. Let $v = (v_0, \ldots, v_d)$, $w = (w_0, \ldots, w_d)$. Acting on v, w, we may arrange that v_0 is in general position and that $(v_0, w_0, w_1, \ldots, w_{d-1})$ is unimodular (cf. 2.4). Say $f_0 v_0 + g_0 w_0 + \cdots + g_{d-1} w_{d-1} = 1$. Adding $(w_d - v_d) f_0 v_0$ to v_d and $(v_d - w_d)(g_0 w_0 + \cdots + g_{d-1} w_{d-1})$ to w_d we reduce to the case that $w_d = v_d$. Now arrange that w_0 is also in general position and add a multiple of $v_0 w_0$ to both v_d and w_d so as to get them in general position (and still equal). Apply the induction hypothesis to

$$(\bar{v}_0, \ldots, \bar{v}_{d-1}), (\bar{w}_0, \ldots, \bar{w}_{d-1}) \in Um_d(R/v_d R, I/v_d R).$$

(3.5) *Remark.* This proof is an important ingredient in the proof of Theorem 3.6.

(3.6) THEOREM. *Let $A = R$, $d \geqslant 2$ (see 2.3). Then $MSE_{d+1}(R, I)$ is an abelian group with the following operation:*

If $mse(v)$, $mse(w) \in MSE_{d+1}(R, I)$, choose representatives $(a_0, a_1, \ldots, a_d) \in mse(v)$, $(b_0, \ldots, b_d) \in mse(w)$ with $a_i = b_i$ for $i \geqslant 1$ (cf. Lemma 3.4), and choose p_0 such that $a_0 p_0 \equiv 1 \bmod (a_1 R + \cdots + a_d R)$. Then

$$mse(w) \cdot mse(v) = mse(a_0(b_0 + p_0) - 1, (b_0 + p_0) a_1, a_2, \ldots, a_d).$$

Remark. For $d=1$ there also is a group structure on $MSE_{d+1}(R,I)$. (See 3.2.) As the next computation shows it can also be characterised by the formula in the theorem. But we do not claim that $MSE_2(R,I)$ is abelian.

(3.7) The proof of this theorem will be finished in 3.32. First let us show that the Theorem is true for $A=R=I$, $d=2$. Say $a_0 p_0 + a_1 p_1 + a_2 p_2 = 1$. We have

$$(b_0\, a_1)\begin{pmatrix} a_0 & a_1 \\ -p_1 & p_0 \end{pmatrix} = (a_0 b_0 - a_1 p_1,\, b_0 a_1 + p_0 a_1)$$
$$= (a_0(b_0+p_0) - 1 + a_2 p_2,\, (b_0+p_0)\, a_1)$$

and the result follows from [17, Thm. 5.2, Cor. 7.4](take transposes).

(3.8) *Remark.* By expanding the techniques used in the remainder of the proof, we might give a proof, independent of [17], that the operation in Theorem 3.6 yields a group. Little would be gained that way, however. Vaserstein's theorem is simply better. It applies to $MSE_3(B)$ for a wider class of rings B, and it gives a good interpretation for the group which one obtains. For instance, we have not found an independent proof for the fact that the group is abelian.

(3.9) For the time being we only consider the absolute case $A=R=I$. Thus we consider $MSE_{d+1}(R,R) = MSE_{d+1}(R)$. Theorem 3.6 will be proved by induction on d. For simplicity we only show how to get from $d=2$ to $d=3$. (The general induction step is essentially the same.) Let $d=3$ now. As the definition of the product in $MSE_{d+1}(R)$ involves choosing representatives, we first construct "composition maps" $*_i$ defined on subsets of $Um_{d+1}(R) \times Um_{d+1}(R)$, taking values in $MSE_{d+1}(R)$. The construction is such that $*_2$ extends $*_1$ and $*_3$ extends $*_2$.

(3.10) DEFINITION OF $*_1$. If $v=(v_0,v_1,v_2,t)$, $w=(w_0,w_1,w_2,t)$ are unimodular and (t) is in general position, choose (z_0,z_1,z_2) so that $mse(\bar v_0,\bar v_1,\bar v_2)\cdot mse(\bar w_0,\bar w_1,\bar w_2) = mse(\bar z_0,\bar z_1,\bar z_2)$ in $MSE_3(R/tR)$ and put $v *_1 w = (z_0,z_1,z_2,t)$. It is easy to show that this is well defined.

(3.11) DEFINITION OF $*_2$. If $v=(v_0,v_1,v_2,v_3)$, $w=(w_0,w_1,w_2,w_3)$, $(v_0,v_1,v_2,w_0,w_1,w_2)$ are unimodular and (p,q), (y,z) are in general position with $p, q \in v_0R+v_1R+v_2R$, $y, z\in w_0R+w_1R+w_2R$, choose (t) in general position such that $t-v_3 \in v_0R+v_1R+v_2R$, $t-w_3\in w_0R+w_1R+w_2R$ (cf. proof of Lemma 3.4) and put $v *_2 w = (v_0,v_1,v_2,t) *_1 (w_0,w_1,w_2,t)$. (We will need p,q,y,z to see that $*_2$ is well-defined. One may also use them at this stage to move t more easily into general position: Use that we may refine the partition of the maximal spectrum in 2.3 so that $V(pR+qR)$ and $V(yR+zR)$ are themselves partitioned by the

V_i that they contain.) To see that $v *_2 w$ is well defined, suppose (t') is in general position, $t' - v_3 \in v_0R + v_1R + v_2R$, $t' - w_3 \in w_0R + w_1R + w_2R$. We have to show that

$$(v_0, v_1, v_2, t) *_1 (w_0, w_1, w_2, t) = (v_0, v_1, v_2, t') *_1 (w_0, w_1, w_2, t').$$

Neither side changes if we operate upon the rows (v_0, v_1, v_2), (w_0, w_1, w_2) by elements of $E_3(R)$. It is easy to see (use a convenient partitioning of the maximal spectrum) that we can arrange (by having elements of $E_3(R)$ act) that (v_0), (v_0, v_1) are in general position. Similarly, arrange that (w_0), (w_0, w_1) are in general position while simultaneously $(v_0, v_1, v_2, w_0, w_1)$ is unimodular. (Look at $(\bar{w}_0, \bar{w}_1, \bar{w}_2) \in Um_3(R/v_0R + v_1R + v_2R)$ and note that we simply want to put $(\bar{w}_0, \bar{w}_1)$ in general position while we are putting (w_0), (w_0, w_1) in general position. These tasks are compatible.) Working on (v_0, v_1, v_2) again, make that (v_0, v_1, w_0, w_1) is unimodular, (v_0), (w_0) both still in general position. Next arrange (cf. proof of Lemma 3.4) that $v_2 = w_2$, (v_2), (v_2, tt') both in general position, $v_1 \equiv w_1 \bmod (v_2, tt')$, dropping the older requirements involving v_0, v_1, w_0, w_1. It is clear from the formulas in Theorem 3.6 (or in 3.7) that

$$(v_0, v_1, v_2, t) *_1 (w_0, w_1, v_2, t) = (v_0, v_1, -t, v_2) *_1 (w_0, w_1, -t, v_2).$$

But the ideals $v_0R + v_1R + v_2R$, $w_0R + w_1R + w_2R$ did not change while we were playing with the v_i, w_i. Therefore the right hand side equals $(v_0, v_1, -t', v_2) *_1 (w_0, w_1, -t', v_2)$ and the rest is clear.

(3.12) DEFINITION OF $*_3$. If $v = (v_0, v_1, v_2, v_3)$, $w = (w_0, w_1, w_2, w_3)$ are unimodular, choose α, β, γ, δ so that there are p, $q \in (v_0 + \alpha v_3)R + (v_1 + \beta v_3)R + v_2R$ with (p, q) in general position and $y, z \in (w_0 + \gamma w_3)R + (w_1 + \delta w_3)R + w_2R$ with (y, z) in general position. (To see that α, β exist, use a partitioning that distinguishes maximal ideals that contain v_2 from those that do not. Note also that p, q exist if and only if $V((v_0 + \alpha v_3)R + (v_1 + \beta v_3)R + v_2R)$ is the union of the empty set $V(I)$ and finitely many subspaces of dimension at most $d - 2$.) Adapt the choice of α, β, γ, δ further, if necessary, to make that $(v_0 + \alpha v_3, v_1 + \beta v_3, v_2, w_0 + \gamma w_3, w_1 + \delta w_3, w_2)$ is unimodular. (As in 3.11 this can be done by adapting γ, δ so that $(\overline{w_0 + \gamma w_3}, \overline{w_1 + \delta w_3})$ is unimodular over $R/(v_0 + \alpha V_3)R + (v_1 + \beta v_3)R + v_2R + w_2R$.) In short, choose $\alpha, \beta, \gamma, \delta$ so that $(v_0 + \alpha v_3, v_1 + \beta v_3, v_2, v_3) *_2 (w_0 + \gamma w_3, w_1 + \delta w_3, w_2, w_3)$ is defined, and define $v *_3 w$ to be the result one gets. As usual we need to show that this result does not depend on the particular choice of $(\alpha, \beta, \gamma, \delta)$. Thus suppose

$$(v_0 + \alpha' v_3, v_1 + \beta' v_3, v_2, v_3) *_2 (w_0 + \gamma' w_3, w_1 + \delta' w_3, w_2, w_3)$$

is also defined. We may add multiples of v_2 to v_0, v_1 and of w_2 to w_0, w_1. Use this to arrange that $(v_0+\alpha v_3, v_1+\beta v_3)$, $(w_0+\gamma w_3, w_1+\delta w_3)$ are in general position and that $(v_0+\alpha v_3, v_1+\beta v_3, w_0+\gamma w_3, w_1+\delta w_3)$ is unimodular (compare with 3.11). Also arrange that these same properties hold with α, β, γ, δ replaced by α', β', γ', δ'. (For this "multiple" goal one may either count as in [6, Sect. 2], in particular 2.11, or observe that one only needs to consider prime ideals $\mathfrak{p}$ with $R/\mathfrak{p}$ infinite; see 2.10.) Now observe that $(v_0+\alpha v_3, v_1+\beta v_3, v_2, v_3) *_2 (w_0+\gamma w_3, w_1+\delta w_3, w_2, w_3)$ can be computed as

$$(v_0+\alpha v_3, v_1+\beta v_3, v_2, t) *_1 (w_0+\gamma w_3, w_1+\delta w_3, w_2, t),$$

where t is chosen such that (t) is in general position, $t-v_3 \in (v_0+\alpha v_3) R+(v_1+\beta v_3) R$, $t-w_3 \in (w_0+\gamma w_3) R+(w_1+\delta w_3) R$. From this one sees that the answer does not change if we add to v_2 an element of $(v_0+\alpha v_3) R+(v_1+\beta v_3) R+tR=v_0R+v_1R+v_3R$. Similarly we may add multiples of w_0, w_1, w_3 to w_2. Therefore we may arrange that $v_2=w_2$, and that (v_2), (tt', v_2) are both in general position (here t' is the analogue of t, obtained when working with α', β', γ', δ' rather than α, β, γ, δ). As before one shows that $(v_0+\alpha v_3, v_1+\beta v_3, v_2, t) *_1 (w_0+\gamma w_3, w_1+\delta w_3, v_2, t)$ equals $(v_0+\alpha v_3, v_1+\beta v_3, -t, v_2) *_1 (w_0+\gamma w_3, w_1+\delta w_3, -t, v_2)$. Hence it equals $(v_0, v_1, -v_3, v_2) *_1 (w_0, w_1, -w_3, v_2)$. (Recall how t has been chosen). But that does not involve α, β, γ, δ any more, so $v *_3 w$ is well-defined.

(3.13) Definition of an Operation in $MSE_{d+1}(R)$ (Case $d=3$, $A=R=I$). If x, $y \in MSE_{d+1}(R)$, choose v, w so that $x=mse(v)$, $y=mse(w)$ and put $x \cdot y = v *_3 w$. We will see later that this is the same operation as in Theorem 3.6 (see 3.32). Now we wish to show that the operation is well-defined. We have to show that $vg *_3 wh = v *_3 w$ for g, $h \in E_{d+1}(R)$. Deduce from the above that the answer does not change if we add a multiple of v_0, v_1 or v_3 to v_2. It also does not change if we add multiples of v_2 to v_0, v_1. If $q \in R$ one may choose α, β, γ, δ so that

$$\begin{gathered}(v_0+\alpha(v_3+qv_2) v_3, v_1+\beta(v_3+qv_2) v_3, v_2, v_3)\\ *_2 (w_0+\gamma w_3, w_1+\delta w_3, w_2, w_3)\end{gathered}$$

is defined. Use this to show that $v *_3 w$ does not change either if we add qv_2 to v_3. Now $E_{d+1}(R)$ is generated by the elements $e_{2*}(*)$ and $e_{*2}(*)$. Therefore the above implies $vg *_3 wh = v *_3 wh$. Similarly $v *_3 wh = v *_3 w$.

(3.14) Lemma. *With the operation of* 3.13, $MSE_{d+1}(R)$ *is an abelian group* (*case* $d=3$, $A=R=I$).

Proof. That it will be abelian is clear from the construction and the fact that the case $d=2$ yields abelian groups. Existence of inverses is easy for the same reason. To get associativity we need to strengthen Lemma 3.4 a bit (cf. [4, Lemma 2.4]). Namely, if $u=(u_0,\dots,u_d)$, $v=(v_0,\dots,v_d)$, $w=(w_0,\dots,w_d)$ are unimodular, we wish to act upon them to achieve $u_i=v_i=w_i$ for $i\geqslant 1$, $(u_1,\dots,u_d)$ in general position. (If we can do that, we can check associativity by reducing to lower dimension. In fact, we can reduce all the way down to MSE_2.) First make that $u_i=v_i$ for $i\geqslant 1$ and make that u_0, v_0 are in general position. Then make that $(u_0v_0, w_0,\dots,w_{d-1})$ is unimodular and exploit that as in 3.4 to make u_d, v_d, w_d equal. Make that w_0 is in general position and add a multiple of $u_0v_0w_0$ to u_d, v_d, w_d to make that moreover u_d $(=v_d=w_d)$ is in general position. Finish by induction.

(3.15) Extend the above to the case $A=R=I$, $d\geqslant 3$.

(3.16) THEOREM (cf. [12, Lemma 2.10; 14, Prop. 1.3]) $(A=R=I)$. *Let* $d\geqslant 2$.

(i) *If* $v=(v_0,\dots,v_d)\in Um_{d+1}(R)$ *then*

$$mse(v_0^2, v_1, v_2,\dots, v_d)=mse(v_0, v_1^2, v_2,\dots, v_d).$$

(ii) *If* $v=(1+at, b_1,\dots,b_d)\in Um_{d+1}(R)$, *then*

$$\begin{aligned} mse(v)&=mse(1+at, b_1t, b_2t, b_3,\dots, b_d)\\ &=mse(1+at, b_1t^2, b_2,\dots, b_d).\end{aligned}$$

(iii) *If* $v=(v_0,\dots,v_d)$, $w=(w_0, v_1,\dots,v_d)$ *are in* $Um_{d+1}(R)$ *then* $mse(w)\cdot mse(v_0^2, v_1,\dots,v_d)=mse(v_0^2w_0, v_1,\dots,v_d)$.

(iv) *If* $v\in Um_{d+1}(R)$, $g\in SL_{d+1}(R)$, *then* $mse(v\cdot g)=mse(v)\cdot mse(g)$.

(3.17) *Remarks.* (1) In contrast with [14, Prop. 1.3] the condition $g\in SL_{d+1}(R)$ can not be replaced by $g\in GL_{d+1}(R)$. (See 4.16, 4.13 below.)

(2) For now we will be working with the description of the group structure by means of $*_3$ (see 3.13).

(3.18) *Proof of Theorem* 3.16.

(i) See [12, Lemma 2.10].

(ii) Adding multiples of $1+at$, b_1t, b_2t *to* b_i $(i\geqslant 3)$ reduces to the case that $(b_3,\dots,b_d)$ is in general position. We may compute in $MSE_3(R/b_3R+\cdots+b_dR)$. As

$$(\bar t, \bar 0)\begin{pmatrix}\bar b_1 & \bar b_2\\ * & *\end{pmatrix}=(\overline{tb_1}\ \overline{tb_2}),$$

we have $mse(\bar{t}, \bar{0}, \overline{1+at}) \cdot mse(\bar{b}_1, \bar{b}_2, \overline{1+at}) = mse(\overline{tb_1}, \overline{tb_2}, \overline{1+at})$ [17, Thm. 5.2], whence the first equality in (ii). To prove the second equality one multiplies $(b_1 t, b_2 t)$ from the right by an element of $E_2(R)$ that modulo $1+at$ looks like $\operatorname{diag}(t, t^{-1})$.

(iii) Say $pv_0 \equiv 1 \bmod(v_1 R \cdots + v_d R)$. Then

$$\begin{aligned} &mse(w) \cdot mse(v_0^2, v_1, \ldots, v_d) \\ &\quad = mse(v_0^2(p^2 + w_0) - 1, (p^2 + w_0)\, v_1, v_2, \ldots, v_d) \\ &\quad = mse(v_0^2(p^2 + w_0) - 1, (v_0(p^2 + w_0))^2\, v_1, v_2, \ldots, v_d) \\ &\quad \underset{\text{(ii)}}{=} mse(v_0^2(p^2 + w_0) - 1, v_1, v_2, \ldots, v_d) = mse(v_0^2 w_0, v_1, \ldots, v_d). \end{aligned}$$

(iv) We argue as in the proof of [14, Prop. 1.3]. As $E_{d+1}(R)$ is normal in $GL_{d+1}(R)$ [15, Cor. 1.4], neither side changes if we replace v by $v\alpha$, g by $\beta g \gamma$, with $\alpha, \beta, \gamma \in E_{d+1}(R)$. Apply Lemma 3.4 to v and the first row of g^{-1} to achieve that this first row of g^{-1} equals $(w_0, v_1, \ldots, v_d)$, where $w_0 \in R$, $(v_1, \ldots, v_d)$ is in general position, $v = (v_0, v_1, \ldots, v_d)$. Say the first row of g is $(u_0, u_1, \ldots, u_d)$ and

$$g^{-1} = \begin{pmatrix} w_0 & v_1 \cdots v_d \\ * & N \end{pmatrix}.$$

Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_q$ be the finitely many maximal ideals containing $J = v_1 R + \cdots + v_d R$ and let S be the complement in R of $\bigcup_{i=1}^q \mathfrak{m}_i$. Note that $J = u_1 R + \cdots + u_d R$ (e.g., use Cramer's rule twice). Let M be the associate matrix of N, such that $NM = MN = \det(N) \cdot 1_d$. As $\det(g) = 1$ we have det $(N) = u_0 \in S$, and det $(M) = u_0^{d-1} \in S$, so $M \in GL_d(S^{-1}R)$. The ring $S^{-1}R$ is semi-local, so

$$M = \begin{pmatrix} u_0^{d-1} & 0 \\ 0 & 1_{d-1} \end{pmatrix} E,$$

with $E \in E_d(S^{-1}R)$. Choose $s \in S$ so that

$$M = \begin{pmatrix} u_0^{d-1} & 0 \\ 0 & 1_{d-1} \end{pmatrix} e_1 \cdots e_r$$

in $GL_d(R[1/s])$, with the e_i elementary (s may very well be a zero-divisor). Note that $\bar{s}$ is invertible in R/J. Choose $p \in R$ with $ps \equiv v_0 - w_0 \bmod J$. As we may add multiples of $v_1, \ldots, v_d$ to v_0 we may arrange that $ps = v_0 - w_0$.

Then

$$\begin{aligned} mse(v\cdot g) &= mse((w_0+v_0-w_0, v_1,\dots,v_d)\cdot g)\\ &= mse(1+(v_0-w_0)u_0, (v_0-w_0)u_1,\dots,(v_0-w_0)u_d)\\ &\underset{(\mathrm{ii})}{=} mse(1+psu_0, (v_0-w_0)u_0^2u_1,\dots,(v_0-w_0)u_0^2u_d). \end{aligned}$$

Now

$$\begin{aligned} &(v_0-w_0)u_0^2(u_1,\dots,u_d)\\ &\quad= (v_0-w_0)u_0(u_1,\dots,u_d)NM\\ &\quad= (v_0-w_0)(-u_0^2)(v_1,\dots,v_d)M \qquad \text{(look at first row of } gg^{-1})\\ &\quad= (w_0-v_0)u_0^2(v_1u_0^{d-1}, v_2,\dots,v_d)e_1\cdots e_r, \end{aligned}$$

where the e_i, however, are in $E_d(R[1/s])$, not $E_d(R)$. On the other hand we may by (ii) multiply coordinates such as $(v_0-w_0)u_0^2u_1$ freely by s^2. Therefore we mimic the effect of the e_i as follows. For each e_i choose a diagonal matrix d_i with positive powers of s^2 on the diagonal, so that $e_id_i = d_ie_i'$ in $GL_d([1/s])$ for some $e_i' \in E_d(R)$. Choose d_i', also a diagonal matrix with positive powers of s^2 on the diagonal, so that d_id_i' is central in $GL_d(R[1/s])$. Then

$$\begin{aligned} &(v_0-w_0)u_0^2(u_1,\dots,u_d)d_1d_1'\cdots d_rd_r'\\ &\quad= (w_0-v_0)u_0^2(v_1u_0^{d-1}, v_2,\dots,v_d)d_1e_1'd_1'\cdots d_re_r'd_r' \end{aligned}$$

(over $R[1/s]$ and one may arrange it to be true over R too) so that

$$\begin{aligned} mse(v\cdot g) &= mse(1+psu_0, (w_0-v_0)v_1u_0^{d-1}, (w_0-v_0)v_2,\dots,(w_0-v_0)v_d)\\ &= mse(1+(v_0-w_0)u_0, ((w_0-v_0)u_0)^{d-1}(w_0-v_0)v_1, v_2,\dots,v_d)\\ &= mse(1+(v_0-w_0)u_0, (w_0-v_0)v_1, v_2,\dots,v_d)\\ &= mse(v)\cdot mse(-u_0, v_1,\dots,v_d) \qquad \text{as} \quad (-u_0)(-w_0)\equiv 1 \bmod J. \end{aligned}$$

But $mse(-u_0, v_1,\dots,v_d)$ is just the inverse of $mse(w_0, v_1,\dots,v_d)$, hence of $mse(g^{-1})$. We have shown that $mse(v\cdot g) = mse(v)\cdot mse(g^{-1})^{-1}$. Substituting $(1, 0,\dots,0)$ for v we see $mse(g) = mse(g^{-1})^{-1}$, and (iv) follows.

(3.19) We now wish to show how the relative case ($R \neq I$) can be reduced to the absolute case. If J is an ideal in the commutative ring B, one may adjoin a unit to J and thus obtain the ring $\mathbb{Z}\oplus J$ with multiplication $(n\oplus i)(m\oplus j) = nm \oplus (nj+mi+ij)$, for $m, n\in\mathbb{Z}$, $i, j\in J$. (Note that $J = 0\oplus J$ is an ideal of $\mathbb{Z}\oplus J$.) In particular, if one does this to the ideal I in R ($R = A$)

one gets a ring $\mathbb{Z} \oplus I$ whose maximal spectrum consists of two parts: One part, viz. $\{\mathfrak{m}: \mathfrak{m} \text{ contains } I\}$, is homeomorphic to the maximal spectrum of $\mathbb{Z}$. The other part, the complement of the first part, is homeomorphic to the complement of $V(I)$ in the maximal spectrum of R. Therefore if $d \geqslant 2$ the maximal spectrum of $\mathbb{Z} \oplus I$ is the union of finitely many subspaces of dimension $\leqslant d$ and we may apply the above results to $MSE_{d+1}(\mathbb{Z} \oplus I)$.

(3.20) *Remark.* It is not true that if R is d-dimensional noetherian, $\mathbb{Z} \oplus I$ must be d-dimensional or noetherian. (Consider, for example, $I = XR + YR$ in $R = \mathbb{Q}[[X, Y]]$.) Thus one needs the setting of 2.3 here. Note that the maximal spectrum of $\mathbb{Z} \oplus I$ is still noetherian, even if the ring $\mathbb{Z} \oplus I$ is not.

(3.21) THEOREM (Excision). *Let $n \geqslant 3$ and let J be an ideal in the commutative ring B. Then the natural maps $MSE_n(\mathbb{Z} \oplus J, J) \to MSE_n(B, J)$, $MSE_n(\mathbb{Z} \oplus J, J) \to MSE_n(\mathbb{Z} \oplus J)$ are bijective.*

Remark. In this theorem $MSE_n(B, J)$ need not be a group.

Proof of the Excision Theorem. Surjectivity is clear for both maps (see [19, Lemma 1]). To see that the second map is injective, consider v, $w \in Um_n(J)$, $g \in E_n(\mathbb{Z} \oplus J)$ with $v \cdot g = w$. Write g as $g_1 g_2$ with $g_1 \in E_n(\mathbb{Z} \oplus J, J)$ and $g_2 \in E_n(\mathbb{Z})$. Clearly $g_2 = \left(\begin{smallmatrix} 1 & 0 \\ * & M \end{smallmatrix}\right)$ with $M \in SL_{n-1}(\mathbb{Z}) = E_{n-1}(\mathbb{Z})$. To see that w is in the same $E_n(\mathbb{Z} \oplus J, J)$ orbit as v, use the following lemma, substituting $\mathbb{Z} \oplus J$ for B.

(3.22) LEMMA. *Let $n \geqslant 3$, $v \in Um_n(J)$. The orbit of v under $E_n(\mathbb{Z} \oplus J, J)$ is invariant under matrices whose transpose is in $E_n^1(B, J)$ (see 2.1, 2.2). It is also invariant under matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & M \end{smallmatrix}\right)$ with $M \in E_{n-1}(B)$.*

Proof. Write $v = (1 + a_1, a_2, \dots, a_n)$, $a_i \in J$, and let $t \in B$. For $2 \leqslant i \leqslant n$, $2 \leqslant j \leqslant n$, $i \neq j$, we have $v \cdot e_{ij}(t) = v \cdot e_{1j}(a_i t)\, e_{ij}(-ta_1)$, which is in v's orbit. (Here one needs that B is commutative.) Also

$$v \cdot e_{i1}(t) = v \cdot e_{ij}(t)\, e_{j1}(1)\, e_{ij}(-t)\, e_{j1}(-1).$$

In the right hand side we may replace $e_{ij}(t)$, $e_{ij}(-t)$, by suitable elements of the normal subgroup $E_n(\mathbb{Z} \oplus J, J)$ of $E_n(\mathbb{Z} \oplus J)$, because of the previous computation. Therefore $e_{i1}(t)$ leaves the orbit invariant and all transposes of "the" generators of $E_n^1(B, J)$ leave the orbit invariant.

(3.23) To finish the proof of the excision theorem, consider u, $w \in Um_n(J)$, $g \in E_n(B, J)$ with $v \cdot g = w$. Note that the transpose of g is in $E_n^1(B, J)$ by Lemma 2.2, and apply Lemma 3.22.

(3.24) *Notation.* We may further write $MSE_n(J)$ for $MSE_n(B, J)$, $n \geqslant 3$, as the excision theorem tells us that the result is independent of the ambient ring B. Note that for $v \in Um_n(J)$ one may view $mse(v) \in MSE_n(J)$ as the orbit of v under $E_n(\mathbb{Z} \oplus J, J)$, and also as the orbit under a larger group of operations on $Um_n(J)$ which contains the operations from Lemma 3.22.

(3.25) Let us now state the relative counterpart of Theorem 3.16, plus some related facts.

THEOREM (cf. [14, Prop. 1.3; 17, Thm. 5.2]) $(A = R,\ d \geqslant 2)$. *$MSE_{d+1}(I)$ enjoys the following properties.*

(i) *If $v = (v_0, \ldots, v_d) \in Um_{d+1}(I)$ then*

$$mse(v_0^2, v_1, \ldots, v_d) = mse(v_0, v_1^2, v_2, \ldots, v_d).$$

(ii) *If $v = (1 + at, b_1, \ldots, b_d) \in Um_{d+1}(I)$ with $a \in I$, $t \in R$, then*

$$mse(v) = mse(1 + at, b_1 t, b_2 t, b_3, \ldots, b_d) = mse(1 + at, b_1 t^2, b_2, \ldots, b_d).$$

(iii) *Let $y = (y_0, \ldots, y_d)$, $v = (v_0, \ldots, v_r, y_{r+1}, \ldots, y_d)$ be I-unimodular, $g \in M_{r+1}(R)$ such that the first row of g is $(y_0, \ldots, y_r)$ and $\overline{\det(g)}$ is a square of a unit in the ring $R/y_{r+1}R + \cdots + y_d R$. Then*

$$mse(v) \cdot mse(y) = mse(z_0, \ldots, z_r, y_{r+1}, \ldots, y_d),$$

where $(z_0, \ldots, z_r) = (v_0, \ldots, v_r) \cdot g$ $(0 \leqslant r \leqslant d)$.

(iv) *Similarly, if $y = (y_0, \ldots, y_d)$, $v = (y_0, \ldots, y_r, v_{r+1}, \ldots, v_d)$ are I-unimodular and $g \in M_{d-r}(R)$ is such that the first row of g is $(y_{r+1}, \ldots, y_d)$ and $\overline{\det(g)}$ is a square of a unit in the ring $R/y_0 R + \cdots + y_r R$, then*

$$mse(v) \cdot mse(y) = mse(y_0, \ldots, y_r, z_{r+1}, \ldots, z_d),$$

where $(z_{r+1}, \ldots, z_d) = (v_{r+1}, \ldots, v_d) \cdot g$ $(0 \leqslant r \leqslant d - 1)$.

(v) *If $v \in Um_{d+1}(I)$ and g is one of the matrices in 3.22 then $mse(v) = mse(v \cdot g)$.*

Remark. If t is a unit in R one easily checks (using part (ii) for instance) that $(v_0, \ldots, v_d) \mapsto (v_0, \ldots, v_{d-1}, tv_d)$ induces a group endomorphism of $MSE_{d+1}(I)$. If t is not a square this endomorphism may be non-trivial (see 4.16, 4.13) so that in (iii), (iv), $\overline{\det(g)}$ really has to be a square of a unit.

(3.26) COROLLARY $(A = R, d \geqslant 2)$ (cf. [14, Cor. 1.4]).

(i) *mse: $SL_{d+1}(R, I) \to MSE_{d+1}(I)$ is a homomorphism.*

(ii) $[SL_{d+1}(R, I), SL_{d+1}(R, I)] \subseteq SL_d(R, I)\, E_{d+1}(R, I)$.

Remark. We will see in 4.16 that $[GL_{d+1}(R, I), GL_{d+1}(R, I)]$ need not be contained in $GL_d(R, I)\, E_{d+1}(R, I)$.

Proof of Corollary. Part (i) follows from part (iii) of the theorem (take $r=d$); (ii) then follows from the fact that $MSE_{d+1}(I)$ is abelian. (Look at $\ker(SL_{d+1}(R, I) \to MSE_{d+1}(I))$).

(3.27) *Proof of Theorem* 3.25(i), (ii), (v). Part (i) follows from 3.16(i) by excision, and (v) is obvious.

(ii) The second equality follows as in 3.18, now using Lemma 3.22.

To prove the first equality observe that

$$\begin{aligned} mse(v) &= mse(1+at, (-at)^2 b_1, b_2, \ldots, b_d) \\ &= mse(1+at, b_1 t^2, b_2, \ldots, b_d) \cdot mse(1+at, a^2, b_2, \ldots, b_d), \end{aligned}$$

where the last equality follows by excision from 3.16(iii). Now observe that $(1+at, a^2)$ is unimodular.

(3.28) Before proving (iii) and (iv) let us introduce a different model for $MSE_n(I)$.

Notation. If J is an ideal in the commutative ring B, $n \geqslant 3$, write $Um_n^1(B, J)$ for the set of $(v_0, \ldots, v_d) \in Um_n(B)$ with $v_0 - 1 \in J$. Write $G_n^1(B, J)$ for the sugroup of $E_n(B)$ generated by $E_n^1(B, J)$ (see 2.1, 2.2) and $\{(\begin{smallmatrix} 1 & 0 \\ 0 & E \end{smallmatrix}) :$ $E \in E_{n-1}(B)\}$.

(3.29) LEMMA. *With n, B, J as above, the natural map $MSE_n(J) \to Um_n^1(B, J)/G_n^1(B, J)$ is bijective.*

Proof. Surjectivity is easy. If v, $w \in Um_n(J)$, $g \in G_n^1(B, J)$ are such that $v \cdot g = w$ then we claim that $mse(v) = mse(w)$. Because $E_n^1(B, J)$ is normal in $G_n^1(B, J)$, we may write $g = hk$ with $h = (\begin{smallmatrix} 1 & 0 \\ 0 & E \end{smallmatrix})$, $E \in E_{n-1}(B)$, $k \in E_n^1(B, J)$. In fact $k \in E_n(B, J)$ by Lemma 2.2. Use Lemma 3.22.

(3.30) *Proof of* 3.25(ii). One extreme is easy: The case $r=0$ follows from 3.16(iii) by excision. At the other extreme, let $r=d$. By part (ii) we may divide y_r, z_r by $\det(g)$ ($=$ square of unit). Therefore assume $\det(g)=1$. We will modify the proof of 3.16(iv) to fit the present needs. First let us change g (and therefore also y) such that the first column of g is also I-unimodular. (One can achieve this by multiplying g from the right with a suitable matrix of the form

$$\begin{pmatrix} 1 & 0 \\ * & 1_d \end{pmatrix}.)$$

Then g normalizes $E_{d+1}^1(R, I)$. To see this, let $(a_0, \ldots, a_d)$ denote the first column of g. Recall that Suslin in [15, Sect. 1] gives a recipe to write $g e_{1i}(t) g^{-1}$ as a product of elementary matrices ($t \in R$, $2 \leqslant i \leqslant d+1$). Say

$d=2$ for simplicity. From the recipe we see that in order to show $ge_{1i}(t)\,g^{-1}\in E^1_{d+1}(R,I)$, it suffices to consider matrices like

$$\begin{pmatrix} 1+a_0a_1p & -a_0^2p & 0\\ a_1^2p & 1-a_0a_1p & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{with } p\in R,$$

and

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1+a_1a_2q & -a_1^2q\\ 0 & a_2^2q & 1-a_1a_2q\end{pmatrix} \quad \text{with } q\in R.$$

There are various ways to see that such matrices are indeed in $E^1_{d+1}(R,I)$. For instance, put

$$Y=\begin{pmatrix} a_0p & -a_0^2p\\ a_1p & -a_0a_1p\end{pmatrix}, \qquad D=\operatorname{diag}(a_1,1).$$

Then $1+DY=e_{12}(a_0)\,e_{21}(a_1p)\,e_{12}(-a_0)$, so that Lemma 2.14 yields $1+YD\in SL_2(R)\cap E^1(R,I)$.

Now that g normalizes $E^1_{d+1}(R,I)$, we represent $mse(v)$, $mse(g)$, $mse(v\cdot g)$ by the corresponding elements of $Um^1_{d+1}(R,I)/G^1_{d+1}(R,I)$ (see Lemma 3.29). Given $\alpha,\beta,\gamma\in E^1_{d+1}(\mathbb{Z}\oplus I,I)$ (which maps into $E^1_{d+1}(R,I)$), we may replace v by $v\alpha$, g by $\beta g\gamma$. Use this (cf. Lemma 3.4 and Prop. 2.4) to reduce to the case: $v=(v_0,\dots,v_d)$, $v_1,\dots,v_{d-1},v_d-1\in I$, $R/v_1R+\cdots+v_dR$ is semi-local, the first row of g^{-1} equals $(w_0,v_1,\dots,v_d)$ for some w_0, the first column of g^{-1} is (necessarily) still I-unimodular. (Taking $v_d\in 1+I$ makes that $V(v_1R+\cdots+v_dR)$ does not intersect $V(I)$, so it eliminates $V(I)$ from the maximal spectrum.)

Let u_i, N, J, S, M, s be as in the proof of 3.16(iv). Multiplication by s induces a bijection of R/J modules I mod $IJ\to I$ mod IJ. Choose $p\in I$ so that $v_0-w_0\equiv ps \bmod IJ$. As we may add I-multiples of $v_1,\dots,v_d$ to v_0, we may arrange that $ps=v_0-w_0$. We find as before that

$$mse(v\cdot g)=mse(1+u_0(v_0-w_0),(w_0-v_0)\,v_1u_0^{d-1},$$
$$(w_0-v_0)\,v_2,\dots,(w_0-v_0)\,v_d).$$

(As it happens, $v\cdot g$ is in $Um_{d+1}(I)$, not just $Um^1_{d+1}(R,I)$, so that the old arguments go through, using Lemma 3.22.) As the first column of g is I-unimodular, we have $u_0w_0\equiv 1 \bmod (v_1I+\cdots+v_dI)$, so that the old arguments show $mse(v\cdot g)=mse(v)\cdot mse$ (first row of $g^{-1})^{-1}$ in $MSE_{d+1}(\mathbb{Z}\oplus I)$. (Each of the three mse's is obtained by lifting via

$Um^1_{d+1}(\mathbb{Z} \oplus I)/G^1_{d+1}(\mathbb{Z} \oplus I)$.) Chase a diagram to check that this means that the same relation holds with the original v, g in $MSE_{d+1}(R, I)$ and finish the proof of 3.25(iii), case $r = d$, as in 3.18.

(3.31) Let $0 < r < d$. We prove part (iii) by induction on d. As modulo $y_{r+1}R + \cdots + y_d R$ the determinant of g is a square of a unit t and $y_0 - 1$ is a multiple of t, we may use (ii) to replace y_1 by $y_1 \det(g)$, and similarly z_1 by $z_1 \det(g)$. Therefore, replacing g by $g \cdot \operatorname{diag}(1, \det(g), 1,..., 1)$, we may further assume that $\det(g)$ is a square in R. Put

$$J = \det(g)\, v_0 R + \cdots + \det(g)\, v_r R + y_{r+1} R + \cdots + y_{d-1} R.$$

By Cramer's rule J is contained in both $y_0 R + \cdots + y_{d-1} R$ and

$$z_0 R + \cdots + z_r R + y_{r+1} R + \cdots + y_{d-1} R.$$

As it is also contained in $v_0 R + \cdots + v_r R + y_{r+1} R + \cdots + y_{d-1} R$, we may add elements of IJ to y_d. Now $(\det(g)\, v_0,..., \det(g)\, v_r, y_{r+1},..., y_d)$ is unimodular, so we may further assume that y_d is in general position. If $d > 2$, go modulo y_d and apply the inductive assumption using the $*_1$ product (see 3.10). If $d = 2$, instead of making $\det(g)$ a square in R, make it 1 modulo y_2 and observe that induction applies again (i.e., note that Vaserstein multiplies by reducing to MSE_2 just as in the $*_1$ product. See [17, Thm. 5.2, Cor. 7.4]; cf. 3.7).

(3.32) Theorem 3.6 follows as in 3.7 from Theorem 3.25(iii), case $r = 1$.

(3.33) *Proof of Theorem* 3.25(iv).

LEMMA ($A = R, d \geqslant 2$). *Let* $p = (p_0,..., p_d)$, $q = (p_0, q_1,..., q_d)$, $r = (p_0, r_1,..., r_d)$, p, q, $r \in Um_{d+1}(I)$, *such that* p_0 *is in general position,* $mse(\bar{p}_1,..., \bar{p}_d) \cdot mse(\bar{q}_1,..., \bar{q}_d) = mse(\bar{r}_1,..., \bar{r}_d)$ *in* $MSE_d(R/p_0 R)$. *Then* $mse(p) \cdot mse(q) = mse(r)$ *in* $MSE_{d+1}(I)$.

Proof. Reduce to the case that $p_i = q_i$ for $i \geqslant 2$. We may assume that $\bar{r}_i$ is computed (in the fashion described in Theorem 3.6) from the p_j, q_j, z_j, where $z = (z_0,..., z_d)$ is *I-unimodular with* $\sum_i q_i z_i = 1$. By excision the result follows from part (iii), applied to $MSE_{d+1}(\mathbb{Z} \oplus I, \mathbb{Z} \oplus I)$.

(3.34) To prove 3.25(iv) first observe that $\det(g) \in I$ and that $\det(g)$ is congruent to a square of an element of I mod$(y_0 R + \cdots + y_r R)$. (Recall that $y_0 - 1 \in I$.) Thus if $r = d - 1$, part (iv) follows from Theorem 3.16(iii) by excision. And if $r < d - 1$, we have

$$mse(y) \cdot mse(y_0,..., y_{d-1}, \det(g)) = mse(y_0,..., y_{d-1}, y_d \cdot \det(g)).$$

But $(y_0,\dots,y_r,\det(g))$ is unimodular, so we simply have

$$mse(y) = mse(y_0,\dots,y_{d-1},y_d\cdot\det(g)).$$

Similarly z_d may be replaced by $z_d\cdot\det(g)$ and we further assume, as in 3.31, that $\det(g)$ is a square in R. Put

$$J = y_1R + \cdots + y_rR + \det(g)\,v_{r+1}R + \cdots + \det(g)\,v_dR.$$

As in 3.31 we may add elements of IJ to y_0 and we may further assume that y_0 is in general position. Use Lemma 3.33 and finish as in 3.31.

(3.35) We finish Section 3 with some remarks. In Lemma 3.29 we obtained a bijection $MSE_n(J)\to Um_n^1(B,J)/G_n^1(B,J)$ from the inclusion map $Um_n(J)\to Um_n^1(B,J)$. There is another way to get a bijection $MSE_n(J)\to Um_n^1(B,J)/G_n^1(B,J)$. Call $v=(v_0,\dots,v_{n-1})\in Um_n(J)$, $w=(w_0,\dots,w_{n-1})\in Um_n^1(B,J)$ *associate* if $\sum v_iw_i=1$. If $v\in Um_n(J)$, it is clear that there is $w\in Um_n^1(B,J)$ associate to it. Conversely, if $w\in Um_n^1(J)$, choose $z\in Um_n(B)$ with $\sum w_iz_i=1$ and note that there is $v\in Um_n(J)$, associate to w, with

$$v=(*,\ z_1(1-w_0),\dots,z_{n-1}(1-w_0)).$$

For $n\geqslant 3$ all elements of $Um_n(J)$ associate to a fixed $w\in Um_n^1(B,J)$ are in the same orbit under $E_n(B,J)$. (Hence under $E_n(\mathbb{Z}\oplus J,J)$ by Lemma 3.22.) To see this, take a closer look at [12] (proof of Cor. 2.8). Similarly, all elements of $Um_n^1(B,J)$ associate to a fixed $v\in Um_n(J)$ are in the same orbit under $E_n^1(B,J)$ $(n\geqslant 3)$. Further, if v is associate to w, then $w\cdot g$ is associate to $v\cdot\operatorname{transpose}(g)^{-1}$ $(g\in G_n^1(B,J))$. Thus we get a bijective correspondence $MSE_n(J)\leftrightarrow Um_n^1(B,J)/G_n^1(B,J)$ with $mse(v)\leftrightarrow$ orbit of w, if v is associate to w. Via this correspondence we can translate results such as Lemma 3.22. We find that $Um_n^1(B,J)/G_n^1(B,J)$ is the same as $Um_n^1(B,J)/E_n^1(B,J)$.

(3.36) Lemma. *Let $v=(v_1,\dots,v_n)$, $w=(w_1,\dots,w_n)$ be associate elements of $Um_n(B)$, $n\geqslant 2$. Then for $k\leqslant n/2$, $mse(v)=mse(w_1,\dots,w_{2k},v_{2k+1},\dots,v_n)$.*

Proof. If $n=2$ then

$$g=\begin{pmatrix} v_1 & v_2\\ -w_2 & w_1\end{pmatrix}\in SL_2(B),$$

so that

$$mse(w_1,w_2)=mse\left((v_1v_2)\,g^{-1}\begin{pmatrix}0&1\\-1&0\end{pmatrix}g\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)=mse(v_1,v_2).$$

(Compare [12, Cor. 2.9].) As $T_2(B) \to T_2(B/v_3B + \cdots + v_nB)$ is surjective, where $T_2(B)$ of course stands for $T_2(B, B)$ in the notation of 3.2, we see that $mse(v) = mse(w_1, w_2, v_3, \ldots, v_n)$. Repeat the argument.

(3.37) If n is even we thus find that a row is in the same orbit as its associate rows. If n is odd the situation is different. If d is even, $A = R$, v, $w \in Um_{d+1}(I)$ are associate, then $mse(w)$ equals

$$mse(v \cdot \operatorname{diag}(1, 1, \ldots, 1, -1))^{-1}.$$

(That this is different will follow from 4.1 and [14, Theorem 5.2], applied to a suitable field.) Compare also with [17, Lemma 13.1].

(3.38) *Exercise* ($A = R, d \geqslant 2$). Show that there is an exact sequence of pointed sets

$$1 \to MSE_{d+1}(I) \to MSE_{d+1}(R \oplus I) \to MSE_{d+1}(R) \to 1.$$

(Cf. [14, Prop. 3.4].)

4. Higher Mennicke Symbols, Stably Free Modules and Cohomotopy Groups

(4.1) Recalll that $A = R$. Let $d \geqslant 2$. If $p_0 \in R$, $a = (a_0, \ldots, a_d)$, $b = (b_0, \ldots, b_d)$, a, $b \in Um_{d+1}(I)$, $a_i = b_i$ for $i \geqslant 1$, $a_0 p_0 \equiv 1 \bmod (a_1R + \cdots + a_dR)$, then $mse(b) \cdot mse(a)^{-1} = mse(1 - (a_0 - b_0)p_0, (a_0 - b_0)a_1, a_2, \ldots, a_d)$. (Use Theorem 3.25(iii) with $r = 1$; cf. 3.7.) It follows that the map $MSE_{d+1}(I) \to MS_{d+1}(R, I)$ given by $mse(v) \mapsto ms(v)$ is a homomorphism. (We use Suslin's notation for higher Mennicke symbols; see [13, Sect. 5]). We claim that the kernel of this homomorphism is equal to the group M generated by the elements of the form $mse(1 + pt, ta_1, \ldots, a_d) \cdot mse(1 + pt, a_1, \ldots, a_d)^{-1}$ with $p \in R$, t, $a_i \in I$. To see this we have to show that the Mennicke relations hold in $MSE_{d+1}(I)/M$. One type of Mennicke relation even holds in $MSE_{d+1}(I)$, by 3.25(v). Remains the multiplicative type. The computation above implies that $mse(a_0, \ldots, a_d) \bmod M$ is multiplicative in a_0. Similarly multiplicativity in a_d, say, will follow if we know that M contains the elements of the form $mse(a_0, \ldots, a_{d-2}, ta_{d-1}, 1 + pt - a_0) \cdot mse(a_0, \ldots, a_{d-1}, 1 + pt - a_0)^{-1}$, with p, t, $a_0 - 1$, $a_1, \ldots, a_{d-1} \in I$. (Use 3.25(iv); cf. 3.33). Indeed M contains these elements, as one sees by adding the last coordinate to the first.

(4.2) THEOREM ($A = R, d \geqslant 1$). *If d is odd, then*

$$GL_{d+1}(R, I) \cap E(R, I) \xrightarrow{mse} MSE_{d+1}(R, I) \to MS_{d+1}(R, I) \to 1$$

is exact. If d is even, then $mse(GL_{d+1}(R, I) \cap E(R, I)) = 1$.

Remarks. For $d = 1$ this is the Bass–Kubota theorem and for $d = 2$, $A = R = I$ it is a special case of a theorem of Vaserstein [17, Cor. 7.4]. We will see in 4.14 that $MSE_{d+1}(R, I) \to MS_{d+1}(R, I)$ need not be injective, even if d is even.

Proof of the theorem. Let $d \geqslant 2$. Recall that $GL_{d+1}(R, I) \cap E(R, I)$ is generated by $E_{d+1}(R, I)$ and elements of the form $(1 + DY)(1 + YD)^{-1}$ with $D = \operatorname{diag}(q, 1, \ldots, 1)$, $q \in A$, $Y \in M_{d+1}(I)$, $1 + DY \in GL_{d+1}(R, I)$ (see Remark 2.9a)). If d is even one sees from Theorem 3.25(ii) that $mse(1 + DY) = mse(1 + YD)$, proving the second half of the theorem.

Let d be odd. It is still clear that $ms(1 + DY) = ms(1 + YD)$, where, as in [14], $ms(g)$ denotes ms (first row of g). It remains to show that

$$mse\colon GL_{d+1}(R, I) \cap E(R, I) \to \ker(MSE_{d+1}(I) \to MS_{d+1}(R, I))$$

is surjective. Therefore, consider $mse(1 + pt, ta_1, a_2, \ldots, a_d)$ with $p \in R$, t, $a_i \in I$, as in 4.1. Choose $r \in a_2 R + \cdots + a_d R$ such that $(1 + pt, a_1, r)$ is unimodular. As $(1 + pt, a_1, r^2)$ is also unimodular, there is $\bar{g} \in SL_2(R/r^2R) \cap E_3(R/r^2R, I/r^2R)$ such that $(\overline{1 + pt}, \bar{a}_1) \cdot g = (\overline{1 + pt}, \overline{a_1 t})$ in $Um_2(R/r^2R, I/r^2R)$. (Use Mennicke symbols or use Lemma 2.14, with $D = \operatorname{diag}(\bar{t}, \bar{1})$, $Y = \left(\begin{smallmatrix} \bar{p} & \bar{a}_1 \\ * & * \end{smallmatrix}\right) \in M_2(I/r^2R)$.)

Let $g \in E_3(R, I)$ be a lift of $\bar{g}$. As d is odd, we may now use the same argument as in Suslin [12, Sect. 2]: Choose $x_2, \ldots, x_d$ and a $(d-1) \times 2$ matrix N so that

$$\begin{pmatrix} x_2 & \cdots & x_d \\ a_2 & \cdots & a_d \end{pmatrix} N = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

(Take

$$N = \begin{pmatrix} a_3 & -x_3 \\ -a_2 & x_2 \\ a_5 & -x_5 \\ -a_4 & x_4 \\ \vdots & \vdots \end{pmatrix}.)$$

Choose a 2×3 matrix M with coefficients in rR so that

$$\begin{pmatrix} 0 & 0 & 1 & x_2 \cdots x_d \\ 1+pt & a_1 & 0 & a_2 \cdots a_d \end{pmatrix} \begin{pmatrix} g & 0 \\ NM & 1_{d-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & x_2 \cdots x_d \\ 1+pt & a_1 t & 0 & a_2 \cdots a_d \end{pmatrix}.$$

Conjugating

$$\begin{pmatrix} g & 0 \\ NM & 1_{d-1} \end{pmatrix} \in E_{d+2}(R, I)$$

by a suitable element of $E_{d+2}(R)$, we obtain $h \in E_{d+2}(R, I)$ with

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1+pt & & a_1 \cdots a_d \end{pmatrix} h = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1+pt & a_1 t & a_2 \cdots a_d \end{pmatrix}.$$

Write $h = \left(\begin{smallmatrix} 1 & 0 \\ * & k \end{smallmatrix}\right)$, with $k \in GL_{d+1}(R, I) \cap E(R, I)$. Then, by Theorem 3.25(iii),

$$mse(1+pt, ta_1, a_2, ..., a_d) \cdot mse(1+pt, a_1, ..., a_d)^{-1} = mse(k).$$

(4.3) Let $d \geqslant 2$. If N is a subgroup of $SL_{d+1}(R, I)$ containing $E_{d+1}(R, I)$, we may by Theorem 3.25(iii) identify $Um_{d+1}(I)/N$ with the abelian group $MSE_{d+1}(I)/mse(N)$. (In fact we may take N somewhat larger still, as one sees from 3.25(iii).) Theorem 4.2 says that for $N = GL_{d+1}(R, I) \cap E(R, I)$ the group $Um_{d+1}(R, I)/N$ is $MS_{d+1}(R, I)$ if d is odd, $MSE_{d+1}(I)$ if d is even. If we take $N = SL_{d+1}(R, I)$, we get the group $Um_{d+1}(I)/SL_{d+1}(R, I)$ in which the non-trivial elements are represented by "non-completable" rows. If d is odd it follows from [12, Thm. 2] that every $d!$ power is trivial in $Um_{d+1}(I)/SL_{d+1}(R, I)$. (Reduce to the case $R = \mathbb{Z} \oplus I$, as usual.) We will see in 4.15 that $Um_{d+1}(I)/SL_{d+1}(R, I)$ need not be a torsion group if d is even.

(4.4) COROLLARY (cf. [14, Prop. 1.5]). *For d odd we have an exact sequence*

$$SL_d(R, I) \to SK_1(R, I) \to MS_{d+1}(R, I) \to Um_{d+1}(R, I)/SL_{d+1}(R, I) \to 1.$$

For d even we have an exact sequence

$$SL_d(R, I) \to SK_1(R, I) \to MSE_{d+1}(R, I) \to Um_{d+1}(R, I)/SL_{d+1}(R, I) \to 1.$$

Proof. Recall $SK_1(R, I) \cong SL_{d+1}(R, I)/GL_{d+1}(R, I) \cap E(R, I)$. The second map in the exact sequence is induced by $mse: SL_{d+1}(R, I) \to MSE_{d+1}(R, I)$.

(4.5) *Remark.* For even d we get the following analogue of [14, Cor. 2.6]. (Notation wt as in loc. cit.) The composite map

$$MSE_{d+1}(R, I) \xrightarrow{wt} SK_1(R, I) \to MSE_{d+1}(R, I)$$

sends $mse(v_0, ..., v_d)$ to $mse(v_0^{d!}, v_1, ..., v_d)$, which is not always the same as $mse(v_0, ..., v_d)^{d!}$, however. (See 4.18 for an example where they differ.)

(4.6) THEOREM (Excision). *Let J be an ideal in the commutative ring B and $n \geqslant 3$. Then the map $MS_n(\mathbb{Z} \oplus J, J) \to MS_n(B, J)$ is an isomorphism.*

Proof. To construct its inverse, use excision for MSE. (See also 3.24.)

(4.7) Recall that if B is a commutative ring $Um_n(B)/GL_n(B)$ is in 1–1 correspondence with the set of isomorphism classes of rank $n-1$ projective modules P with $P \oplus B$ free. In particular, if $A = R = I$, $Um_{d+1}(R)/GL_{d+1}(R)$ is in 1–1 correspondence with the set of isomorphism classes of rank d stably free projective modules. (Use stability for K_0; see [2, Ch. IV Cor. 3.5].) If d is odd, then $Um_{d+1}(R)/SL_{d+1}(R) = Um_{d+1}(R)/GL_{d+1}(R)$, because Mennicke relations hold in the left hand side. (For a better reason, see 4.9 below.) Therefore let us look at $Um_{d+1}(R)/SL_{d+1}(R)$ for even d. It is acted upon by $GL_1(R)$. If this action is non-trivial (for an example see 4.16), then we do not get a group structure on $Um_{d+1}(R)/GL_{d+1}(R)$ (at least not one compatible with the group structure on $Um_{d+1}(R)/SL_{d+1}(R)$). Thus from our point of view $Um_{d+1}(R)/SL_{d+1}(R)$ is to be preferred over $Um_{d+1}(R)/GL_{d+1}(R)$. We interpret it as follows.

If B is commutative, $n \geqslant 2$, $Um_n(B)/SL_n(B)$ is in 1–1 correspondence with the set of isomorphism classes of *oriented* rank $n-1$ projectives P with $P \oplus B$ free. Here an orientation is a generator of the free rank 1 module $\bigwedge^{n-1} P$. The correspondence goes like this: If $v \in Um_n(B)$, then v determines a split injection $B \to B^n$, whose cokernel is a projective module P. Using a splitting $B^n \to B$ of v we get an isomorphism $B^n \to P \oplus B$, hence an isomorphism $\bigwedge^n B^n \to \bigwedge^{n-1} P$, and the image of $e_1 \wedge \cdots \wedge e_n$ is an orientation of P. Thus we have

(4.8) THEOREM ($A = R = I, d \geqslant 2$). *The set of isomorphism classes of oriented stably free rank d projective modules carries the structure of an abelian group. If d is odd it is also the set of isomorphism classes of stably free rank d projective modules (without orientation).*

(4.9) *Remark.* If P is an oriented stably free projective module with a rank 1 free direct summand, then $GL(P)$ obviously acts transitively on the set of orientations of P. Therefore, if n is even, $Um_n(B)/SL_n(B) = Um_n(B)/GL_n(B)$, by a lemma of Bass [3, Cor. 4.2].

(4.10) Now let R be a noetherian ring of dimension d ($d \geqslant 2$) which is at the same time a topological ring such that the group of units R^* in R is open and $u \mapsto u^{-1}$ is continuous on R^*. (Thus R satisfies condition (1) of Swan [18, Sect. 1].)

LEMMA. *The orbits of $Um_{d+1}(R)$ under $E_{d+1}(R)$ are open.*

Remark. For this lemma it is not essential that R is noetherian of dimension d. Condition (1) of Swan [18, Sect. 1] will do. To see this one has to make the proof that follows more explicit.

Proof of Lemma. Let $v = (v_0, ..., v_d)$ be unimodular and let $w = (w_0, ..., w_d)$ be sufficiently close to it. Put $v_j^{(r)} = v_j$ for $j \leqslant r$, $v_j^{(r)} = w_j$ for $j > r$. If $\Sigma v_i p_i = 1$, then $\alpha_r = \Sigma v_i^{(r)} p_i$ is close to 1 and hence invertible. We get $mse(v^{(r-1)}) \cdot mse(v^{(r)})^{-1} =_{4.1} mse(*, 1-(v_r - w_r) p_r \alpha_r^{-1}, *) = 1$, as $1-(v_r - w_r) p_r \alpha_r^{-1}$ is close to 1 and hence invertible. Thus $mse(w) = mse(v^{(d-1)}) = \cdots = mse(v)$.

(4.11) Let X be a finite simplicial complex of dimension d, $d \geqslant 2$, and let R be a dense subring of the ring $\mathbb{R}^X$ of continuous real valued functions on X, satisfying the conditions in 4.10. (see Swan [18, Thm. 6.3] for the construction of such R.) Lemma 4.10 shows

THEOREM. $MSE_{d+1}(R) \cong [X, \mathbb{R}^{d+1} \setminus \{0\}]$, *the set of homotopy classes of continuous maps* $X \to \mathbb{R}^{d+1} \setminus \{0\}$.

Remarks. (1) Using the action of $SL_{d+1}(\mathbb{R}) = E_{d+1}(\mathbb{R})$ on $\mathbb{R}^{d+1}$, one sees that it does not matter whether one takes free homotopy classes or spaces with base points. (The base point of $\mathbb{R}^{d+1} \setminus \{0\}$ would be $(1, 0, ..., 0)$.)

(2) Of course $[X, \mathbb{R}^{d+1} \setminus \{0\}]$ is the same as $[X, S^d]$. (Here S^d is the d-sphere.) As for most values of d there is no suitable way to multiply the two projection maps $S^d \times S^d \to S^d$ in $[S^d \times S^d, S^d]$ (see Adams [1, Thm. 1.1(a)]) we see that some restriction on the dimension of R is necessary in Theorem 3.6. In contrast, no such restriction is needed in Suslin's result [12, Sect. 2] that the rows $(a_1^{m_1}, ..., a_r^{m_r})$ and $(a_1^{m_1 \cdots m_r}, a_2, ..., a_r)$ are in the same orbit under $E_{r+1}(B) \cap SL_r(B)$, if $(a_1, ..., a_r) \in Um_r(B)$, $r \leqslant 4$, and at least one of $r, m_1, ..., m_r$ is even ($m_i \geqslant 1$).

(4.12) As $d \geqslant 2$, we are in the stable range of the suspension theorem [11, Ch. 8, Sect. 5, Thm. 11], i.e., the suspension map $[X, S^d] \to [SX, S^{d+1}]$ is bijective. As $[SX, S^{d+1}]$ is an abelian group, we may ask if the (bijective) map $MSE_{d+1}(R) \to [SX, S^{d+1}]$ is a homomorphism. This is indeed the case.

Algebraically the map can be imitated by the homomorphism $MSE_{d+1}(R) \to MSE_{d+1}(R')$, sending $mse(v_0,\ldots,v_d)$ to $mse(1-2t, 4(t-t^2)v_0,\ldots,4(t-t^2)v_d)$, where R' is a suitable subring of $\mathbb{R}^{SX}$ (such that $MSE_{d+1}(R') \cong [SX, S^{d+1}]$) and t is the suspension parameter $(0 \leqslant t \leqslant 1)$. We have to show that the identification of $MSE_{d+1}(R')$ with $[SF, S^{d+1}]$ respects group structure. This follows from the fact that the two group structures "commute" in the sense of [5, Proof of Prop. 9.9]. Thus $MSE_{d+1}(R)$ can be interpreted as a cohomotopy group, and also as $H^d(X, \mathbb{Z})$ by [23, Ch. V, Cor. 6.19].

(4.13) Let us specialize further, taking

$$X = S^d = \{(x_0,\ldots,x_d) \in \mathbb{R}^n : \textstyle\sum x_i^2 = 1\}.$$

Take $R = R_d = T^{-1}\mathbb{R}[X_0,\ldots,X_d]/(\Sigma X_i^2 - 1)$, where T is the multiplicative set of polynomial functions that do not have any zero on S^d. We get $MSE_{d+1}(R_d) \cong \mathbb{Z}$, via the topological degree (i.e., $mse(v) \leftrightarrow$ degree of the map $x \mapsto v(x)/\|v(x)\|$). As generator of $MSE_{d+1}(R_d)$ we may take $mse(x_0,\ldots,x_d)$. We can make several instructive computations. From [23, Ch IV 9.2, 10.6, 10.8 (and the remarks preceding 10.8)] we see

LEMMA. (i) *If d is odd, $d \neq 1, 3, 7$, then $mse(SL_{d+1}(R_d))$ consists of the elements of even degree.*

(ii) *If $d = 1, 3, 7$ then $mse(SL_{d+1}(R_d))$ is all of $MSE_{d+1}(R_d)$.*

(iii) *If d is even, then $mse(SL_{d+1}(R_d)) = 1$.*

(4.14) The element $mse(x_0,\ldots,x_d)^2$ is a non-trivial element of the kernel of $MSE_{d+1}(R_d) \to MS_{d+1}(R_d)$ because it goes to $ms(x_0^2, x_1,\ldots,x_d) = ms(1, x_1,\ldots,x_d) = 1$. For d odd, $d \neq 1, 3, 7$, it follows from this and from Lemma 4.13(i), Theorem 4.2, that $SL_{d+1}(R_d) \to MSE_{d+1}(R_d) \to MS_{d+1}(R_d) \to 1$ is exact. Thus in this case $MS_{d+1}(R_d) \cong \mathbb{Z}/2\mathbb{Z}$ with generator $ms(x_0,\ldots,x_d)$. In other cases we will get the same description. (We still have to see that $ms(x_0,\ldots,x_d)$ is non-trivial in the other cases). It is easy to see that one has for each $d, d \geqslant 1$, a homomorphism $MS_{d+1}(R_d) \to MS_{d+2}(R_{d+1})$ sending $ms(\bar{f}_0,\ldots,\bar{f}_d)$ to $ms(f_1,\ldots,f_d \cdot x_{d+1})$, where $\bar{f}_i$ equals f_i modulo $x_{d+1}R_{d+1}$. (We identify $R_{d+1} \bmod x_{d+1}R_{d+1}$ with R_d. This corresponds with the equatorial embedding of S^d in S^{d+1}. Compare also 4.12.) Observe that the generator $ms(x_0,\ldots,x_d)$ is sent to the generator $ms(x_0,\ldots,x_{d+1})$. As for large odd d the generator is non-trivial it must always be non-trivial. [The referee suggests to prove this by showing that one has a homomorphism $MS_{d+1}(R_d) \to \mathbb{Z}/2\mathbb{Z}$ via the notion of the degree mod 2 of a smooth map $S^d \to S^d$. Recall that this degree mod 2 is obtained by taking a non-critical value of the smooth map and counting the number of inverse images of the chosen value modulo 2. By Sard's theorem the set of non-critical values has measure 1. Let us show that the degree mod 2 satisfies the multiplicative Mennicke relation. Thus consider $(f_0,\ldots,f_d)$, $(g_0, f_1,\ldots,f_d) \in Um_{d+1}(R_d)$ and let $sph(f_0,\ldots,f_d)$ denote the map $S^d \to S^d$ corresponding with $(f_0,\ldots,f_d)$. If there is a noncritical value of the form $(0, c_1,\ldots, c_d)$ for $sph(f_0,\ldots,f_d)$, $sph(g_0, f_1,\ldots,f_d)$, $sph(f_0 g_0, f_1,\ldots,f_d)$ simultaneously, then it is easy. In general we may achieve that some $(0, c_1,\ldots, c_d)$ is a critical value for both $sph(f_0,\ldots,f_d)$ and $sph(g_0, f_1,\ldots,f_d)$, with the two inverse images of $(0, c_1,\ldots, c_d)$ disjoint from each other, by adding a real linear combination of $f_1,\ldots,f_d$ to f_0 and also a real linear combination of $f_1,\ldots,f_d$ to g_0. By multiplying f_0, g_0 by positive real numbers we can make $(0, c_1,\ldots, c_d)$ noncritical for $sph(f_0 g_0, f_1,\ldots,f_d)$ too.] Summing up, we have $MS_{d+1}(R_d) \cong \mathbb{Z}/2\mathbb{Z}$ for all $d \geqslant 1$. But the sequence $SL_{d+1}(R_d) \to MSE_{d+1}(R_d) \to MS_{d+1}(R_d)) \to 1$ is exact only for d odd, $d \neq 1, 3, 7$. (Use 4.13.)

(4.15) We now describe the group $Um_{d+1}(R_d)\,/SL_{d+1}(R_d)$ of Theorem 4.8 (see 4.3). For even d it equals $MSE_{d+1}(R_d) \cong \mathbb{Z}$, by 4.13. For odd d, $d \neq 1$, 3, 7 it equals $MS_{d+1}(R_d) \cong \mathbb{Z}/2\mathbb{Z}$, by 4.13 and 4.14. For $d = 1$, 3, 7 it vanishes. (Use 4.13(ii)).

(4.16) The element $D = \mathrm{diag}(-1, 1,\ldots, 1)$ acts by sending $mse(x_0,\ldots, x_d)$ to $mse(-x_0, x_1,\ldots, x_d) = mse(x_0,\ldots, x_d)^{-1}$. So it sends any element to its inverse. Now let d be odd and let $g \in SL_{d+1}(R_d)$ be chosen such that $mse(g) \neq 1$. (Use Lemma 4.13(i), (ii)). As $d+1$ is now even it is easy to check that $mse(DgD^{-1})$ equals $mse(g)^{-1}$, hence not $mse(g)$. We find that $DgD^{-1}g^{-1} \notin GL_d(R_d)\,E_{d+1}(R_d)$. (Compare 3.26(ii)). On the other hand one derives from [9, Sect. 7] that $SL_{d+1}(R_d)/E_{d+1}(R_d) \cong \pi_d(SL_{d+1}(\mathbb{R}))$. Thus

PROPOSITION. *For d odd, the group $SL_{d+1}(R_d)/E_{d+1}(R_d)$ is abelian, but $GL_{d+1}(R_d)/E_{d+1}(R_d)$ is not.*

Remark. To get examples where $SL_m(R)/E_m(R)$ is not abelian, $m \geqslant 3$, take $m = 2n$, $n \neq 1$, 2 or 4, $R = \mathbb{R}^X$, $X = S^{2n-1} \times S^{2n-1}$. (By Milnor [9, Sect. 7] one may compute in $[X, SO(2n)]$. By [22, Ch. X, Sect. 5] it suffices to check that a certain Samelson product is non-zero. By [22, Ch. X, Thm. 7.10] it is such a product whose order is determined in [10].)

(4.17) The multiplicative relations fail rather badly in $MSE_{d+1}(R_d)$ as the following example shows. (This was pointed out by C. Weibel.)

EXAMPLE (cf. Weibel [22, Example 2.2(c)]).

$$mse(x_1(2x_0 + x_1), x_0^2 + x_0x_1, x_2,\ldots, x_d) = mse(x_0, x_1,\ldots, x_d)^{-2} \neq 1,$$

but $mse(x_1, x_0^2 + x_0x_1, x_2,\ldots, x_d)$ and

$$mse(2x_0 + x_1, x_0^2 + x_0x_1, x_2,\ldots, x_d) = mse(2x_0 + x_1, -x_0^2, x_2,\ldots, x_d)$$

are both trivial ($d \geqslant 2$). This example also works in

$$MSE_{d+1}(\mathbb{Q}[X_0, \dots, X_d]/(\Sigma X_i^2 - 1)).$$

To see this, note for instance that

$$\begin{aligned}(x_1(2x_0 + x_1), x_0^2 + x_0 x_1)&\begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix}^2 \\ &\equiv ((-1 - 2x_0 x_1)\, x_1^2, 2x_0^2 x_1^2 + x_0^2 + x_0 x_1) \operatorname{mod}(x_0^2 + x_1^2 - 1),\end{aligned}$$

while

$$\begin{aligned}&mse(x_1^2, 2x_0^2 x_1^2 + x_0^2 + x_0 x_1, x_2, \dots, x_d) \\ &\quad = mse(x_1, 2x_0^2 x_1^2 + x_0^2 + x_0 x_1, x_2^2, x_3, \dots, x_d) = 1,\end{aligned}$$

so that

$$\begin{aligned}&mse((-1 - 2x_0 x_1)\, x_1^2, 2x_0^2 x_1^2 + x_0^2 + x_0 x_1, x_2, \dots, x_d) \\ &\quad = mse(-1 - 2x_0 x_1, 2x_0^2 x_1^2 + x_0^2 + x_0 x_1, x_2, \dots, x_d) \\ &\quad = mse(-1 - 2x_0 x_1, x_0^2, x_2, \dots, x_d) \\ &\quad = mse(-1 - 2x_0 x_1, x_0, x_2^2, \dots, x_d) = 1.\end{aligned}$$

(4.18) Of course one gets from 4.14 a simpler example of the failure of multiplicative relations in $MSE_{d+1}(R_d)$: If n is an integer, $n \geqslant 1$ then $mse(x_0^{2n}, x_1, \dots, x_d) = mse(x_0^2, x_1, \dots, x_d)^n = mse(1, x_1, \dots, x_d)^n = 1$, but $mse(x_0, \dots, x_d)^{2n} \neq 1$.

(4.19) We now give an example showing that the analogue of Suslin [14, Lemma 1.1] fails for MSE.

Example. Let

$$M = \begin{pmatrix} 2x_1^2 - 1 & 2x_0 x_1 \\ 2x_1 x_2 - 2x_0 x_1 & 2x_0 x_2 - 2x_0^2 + 1 \end{pmatrix}.$$

Then $\bar{M} \in SL_2(R_2/x_2 R_2)$ and $(x_0^2 + x_0 x_2 + x_1^2, x_1 x_2)\, M = (x_1^2 - x_0 x_2 - x_0^2, 2x_0 x_1 + x_1 x_2)$ so that

$$\begin{aligned}&mse(x_1^2 - x_0 x_2 - x_0^2, 2x_0 x_1 + x_1 x_2, x_2) \\ &\quad = mse(x_0^2 + x_0 x_2 + x_1^2, x_1 x_2, x_2)\, mse(2x_1^2 - 1, 2x_0 x_1, x_2) \\ &\quad = mse(2x_1^2 - 1, 2x_0 x_1, x_2).\end{aligned}$$

Also,

$$\begin{pmatrix} x_1 & x_0 \\ -x_0 & x_1 \end{pmatrix}^2 = \begin{pmatrix} x_1^2 - x_0^2 & 2x_0x_1 \\ * & * \end{pmatrix}$$

so that

$$mse(x_1^2 - x_0x_2 - x_0^2, 2x_0x_1 + x_1x_2, x_2) = mse(x_1, x_0, x_2)^2 \neq 1.$$

On the other hand, $(x_1^2 - x_0x_2 - x_0^2)R_2 + (2x_0x_1 + x_1x_2)R_2 = (x_0^2 + x_0x_2 + x_1^2)R_2 + x_1x_2R_2$ has height 2 in R. (It is the intersection of four maximal ideals in R_2.) Therefore by Suslin [14, Lemma 1.1] (cf. [13, Lemma 5.3, Lemma 5.4]) one has $ms(x_1^2 - x_0x_2 - x_0^2, 2x_0x_1 + x_1x_2, x_2) = ms(x_0^2 + x_0x_2 + x_1^2, x_1x_2, x_2) = 1$. Thus the analogue of Suslin [14, Lemma 1.1] fails for MSE. For MSE the correct rule is given by Theorem 3.25(iv), which is of course also valid for MS.

(4.20) *Questions.* (1) If B is a commutative ring and $n \geqslant 3$ is such that $MSE_{n+1}(B)$ vanishes, is there a group structure on $MSE_n(B)$ satisfying the properties in Theorem 3.25? (Cf. [17, Thm. 5.2, Cor. 7.4].)

(2) Similarly, if R is d-dimensional and $d \leqslant 2n - 4$, do we have such a group structure on $MSE_n(R)$? ($2n - 4$ is the bound suggested by the Suspension Theorem; cf. 4.12.)

(3) Can one give interpretations for $MSE_{d+1}(R)$ for $d > 2$, similar to the one given by Vaserstein for $d = 2$; e.g., in the line of [12, Sect. 5]?

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