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FINITE SCHUR FILTRATION DIMENSION FOR MODULES OVER AN ALGEBRA WITH SCHUR FILTRATION

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Abstract. Let $G = \mathrm{GL}_N$ or SL_N as reductive linear algebraic group over a field k of characteristic $p > 0$. We prove several results that were previously established only when $N \leqslant 5$ or $p > 2^N$: Let G act rationally on a finitely generated commutative k-algebra A and let $\mathrm{gr}\, A$ be the Grosshans graded ring. We show that the cohomology algebra $H^*(G, \mathrm{gr}\, A)$ is finitely generated over k. If moreover A has a good filtration and M is a Noetherian A-module with compatible G action, then M has finite good filtration dimension and the $H^i(G, M)$ are Noetherian A^G-modules. To obtain results in this generality, we employ functorial resolution of the ideal of the diagonal in a product of Grassmannians.

1. Introduction

Consider a connected reductive linear algebraic group G defined over a field k of positive characteristic p. We say that G has the cohomological finite generation (CFG) property if the following holds: Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. (So G acts from the right on $\mathrm{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a k-algebra. Here, as in [13, I.4], we use the cohomology introduced by Hochschild, also known as 'rational cohomology'.

The aim of this paper is to take one more step towards proving the conjecture that every reductive linear algebraic group has the CFG property. The proof will be finished by Touzé [19]. The key point of the present paper is to remove restrictions on the characteristic from [23].

Our proofs use resolution of the diagonal in products of Grassmannians. Thus they apply only to the groups SL_N, GL_N. But recall ([21], [22], [23]) that for the conjecture these cases suffice. Also recall that the conjecture implies the main results of this paper, as well as their analogues for other reductive groups.

To formulate the main results, let $N \geqslant 1$ and let G be the connected reductive

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linear algebraic group GL_N or SL_N over an algebraically closed field k of characteristic $p > 0$. Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. Let M be a Noetherian A-module on which G acts compatibly. This means that the structure map $A \otimes M \to M$ is a G-module map. Our main theorem is

Theorem 1.1. *If A has a good filtration, then M has finite good filtration dimension and each $H^i(G, M)$ is a Noetherian A^G-module.*

One may also formulate the first part in terms of polynomial representations of GL_N. Recall that a finite-dimensional (as k vector space) rational representation of GL_N is called polynomial if it extends to the monoid of N by N matrices without poles along the locus where the determinant vanishes. Unlike Green [9] we cannot restrict ourselves to finite-dimensional representations, so we define a representation to be polynomial if it is a union of finite-dimensional polynomial representations. In other words, we allow infinite-dimensional comodules for the bialgebra of regular functions on the monoid.

So let A be a finitely generated commutative k-algebra on which GL_N acts polynomially by k-algebra automorphisms. Let M be a Noetherian A-module on which GL_N acts compatibly and polynomially.

Theorem 1.2. *If A has Schur filtration, then M has finite Schur filtration dimension.*

Remark 1.3. The $H^i(\mathrm{GL}_N, M)$ are less interesting now, because the part of nonzero polynomial degree in M does not contribute to $H^i(\mathrm{GL}_N, M)$.

Now let A be a finitely generated commutative k-algebra on which SL_N acts rationally by k-algebra automorphisms. One then has a Grosshans graded algebra $\mathrm{gr}\, A$ and we can remove the restrictions on the characteristic in [21, Theorem 1.1].

Corollary 1.4. *The k-algebra $H^*(\mathrm{SL}_N, \mathrm{gr}\, A)$ is finitely generated.*

The method of proof of the main result is based on the functorial resolution [16] of the diagonal of $Z \times Z$ when Z is a Grassmannian of subspaces of k^N. This is used inductively to study equivariant sheaves on a product X of such Grassmannians. That leads to a special case of the theorems, with A equal to the Cox ring of X, multigraded by the Picard group $\mathrm{Pic}(X)$, and M compatibly multigraded. Next one treats cases when on the same A the multigrading is replaced with a 'collapsed' grading with smaller value group and M is only required to be multigraded compatibly with this new grading. Here the trick is that an associated graded of M has a multigrading that is collapsed a little less. The suitably multigraded Cox rings now replace the 'graded polynomial algebras with good filtration' of [21] and the method of [23] applies to finish the proof of Theorem 1.1. Then Corollary 1.4 follows in the manner of [21].

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2. Recollections and conventions

Some unexplained notations, terminology, properties, ... can be found in [13].

From now on, with the exception of Section 8, we put $G = \mathrm{GL}_N$, with B^+ its subgroup of upper triangular matrices, B^- the subgroup of lower triangular matrices, $T = B^+ \cap B^-$ the diagonal subgroup, $U = U^+$ the unipotent radical of B^+. The roots of U are positive. The character group $X(T)$ has a basis $\varepsilon_1 \dots, \varepsilon_N$ with $\varepsilon_i(\mathrm{diag}(t_1, \dots, t_N)) = t_i$. An element $\lambda = \sum_i \lambda_i \varepsilon_i$ of $X(T)$ is often denoted $(\lambda_1, \dots, \lambda_N)$. It is called a polynomial weight if the λ_i are nonnegative. It is called a dominant weight if $\lambda_1 \geqslant \cdots \geqslant \lambda_N$. It is called antidominant if $\lambda_1 \leqslant \cdots \leqslant \lambda_N$. The fundamental weights $\varpi_1, \dots, \varpi_N$ are given by $\varpi_i = \sum_{j=1}^{i} \varepsilon_j$.

If $\lambda \in X(T)$ is dominant, then $\mathrm{ind}_{B^-}^{G}(\lambda)$ is the *dual Weyl module* or *costandard module* $\nabla_G(\lambda)$, or simply $\nabla(\lambda)$, with highest weight λ. The *Grosshans height* of λ is $\mathrm{ht}(\lambda) = \sum_i (N - 2i + 1)\lambda_i$. It extends to a homomorphism $\mathrm{ht} : X(T) \otimes \mathbb{Q} \to \mathbb{Q}$. The determinant representation has weight ϖ_N and one has $\mathrm{ht}(\varpi_N) = 0$. Each positive root β has $\mathrm{ht}(\beta) > 0$. If λ is a dominant polynomial weight, then $\nabla_G(\lambda)$ is called a *Schur module*.

If α is a partition with at most N parts then we may view it as a dominant polynomial weight and the *Schur functor* S^α maps $\nabla_G(\varpi_1)$ to $\nabla_G(\alpha)$. (This is the convention followed in [16]. In [1] the same Schur functor is labeled with the conjugate partition $\tilde{\alpha}$. See also [9, Theorem (4.8f), 5.6].) The formula $\nabla(\lambda) = \mathrm{ind}_{B^-}^{G}(\lambda)$ just means that $\nabla(\lambda)$ is obtained from the Borel-Weil construction: $\nabla(\lambda)$ equals $H^0(G/B^-, \mathcal{L}_\lambda)$ for a certain line bundle $\mathcal{L}_\lambda$ on the flag variety G/B^-. There are similar conventions for SL_N-modules. For instance, the costandard modules for SL_N are the restrictions of those for GL_N.

The Grosshans height on $X(T)$ induces one on $X(T \cap \mathrm{SL}_N) \otimes \mathbb{Q}$. The *multicone* $k[\mathrm{SL}_N/U]$ consists of the f in the coordinate ring $k[\mathrm{SL}_N]$ that satisfy $f(xu) = f(x)$ for $u \in U \cap \mathrm{SL}_N$. As an SL_N-module it is the direct sum of all costandard modules. It is also a finitely generated algebra [14], [10].

Definition 2.1. A *good filtration* of a G-module V is a filtration $0 = V_{\leqslant -1} \subseteq V_{\leqslant 0} \subseteq V_{\leqslant 1} \cdots$ by G-submodules $V_{\leqslant i}$ with $V = \bigcup_i V_{\leqslant i}$, so that its associated graded $\mathrm{gr}\, V$ is a direct sum of costandard modules.

A *Schur filtration* of a polynomial GL_N-module V is a filtration $0 = V_{\leqslant -1} \subseteq V_{\leqslant 0} \subseteq V_{\leqslant 1} \cdots$ by GL_N-submodules with $V = \bigcup_i V_{\leqslant i}$, so that its associated graded $\mathrm{gr}\, V$ is a direct sum of Schur modules.

The *Grosshans filtration* of V is the filtration with $V_{\leqslant i}$ the largest G-submodule of V whose weights λ all satisfy $\mathrm{ht}(\lambda) \leqslant i$. Good filtrations and Grosshans filtrations for SL_N-modules are defined similarly.

The literature contains more restrictive definitions of good/Schur filtrations. Ours are the right ones when dealing with infinite-dimensional representations [20], cf. [13, II.4.16, Remark 1].

Proposition 2.2. *Let V be a polynomial representation of GL_N. The following are equivalent:*

(1) *V has a good filtration.*
(2) *V has a Schur filtration.*

(3) *The Grosshans filtration of* V *is a Schur filtration.*
(4) *The restriction* $\mathrm{res}_{\mathrm{SL}_N}^{\mathrm{GL}_N} V$ *has a good filtration.*
(5) *The Grosshans filtration of the restriction* $\mathrm{res}_{\mathrm{SL}_N}^{\mathrm{GL}_N} V$ *is a good filtration.*
(6) $H^1(\mathrm{SL}_N, k[\mathrm{SL}_N/U] \otimes V) = 0$.

Proof. (3) ⇒ (2) ⇒ (1) ⇒ (4) ⇒ (6) ⇒ (5) is well known [13, II 4.16 and proof of A.5], compare [20, Exer. 4.1.3]. Now assume 5. We may decompose V into weight spaces (also known as polynomial degrees) for the center of G. One may replace V by one of these weight spaces. The Grosshans filtration of $\mathrm{res}_{\mathrm{SL}_N}^{\mathrm{GL}_N} V$ is then a good filtration which may be reinterpreted as a Schur filtration on V. □

Definition 2.3. If V is a GL_N-module, and $m \geqslant -1$ is an integer so that $H^{m+1}(\mathrm{SL}_N, k[\mathrm{SL}_N/U] \otimes \mathrm{res}_{\mathrm{SL}_N}^{\mathrm{GL}_N} V) = 0$, then we say that V has *good filtration dimension* at most m. (Compare [7].) The case $m = 0$ corresponds with V having a good filtration. And for $m \geqslant 0$ it means that V has a resolution

$$0 \to V \to N_0 \to \cdots \to N_m \to 0$$

in which the N_i have good filtration. We say that V has good filtration dimension precisely m, notation $\dim_\nabla(V) = m$, if m is minimal so that V has good filtration dimension at most m. In that case $H^{i+1}(\mathrm{SL}_N, k[\mathrm{SL}_N/U] \otimes \mathrm{res}_{\mathrm{SL}_N}^{\mathrm{GL}_N} V) = 0$ for all $i \geqslant m$. In particular $H^{i+1}(G, V) = 0$ for $i \geqslant m$. If there is no finite m so that $\dim_\nabla(V) = m$, then we put $\dim_\nabla(V) = \infty$. Similar definitions apply to SL_N-modules.

If V is a polynomial representation then $\dim_\nabla(V)$ is also called the Schur filtration dimension. Indeed if for such V one has $\dim_\nabla(V) \leqslant m$, $m \geqslant 0$, then V has a resolution

$$0 \to V \to N_0 \to \cdots \to N_m \to 0$$

in which the N_i have Schur filtration.

3. Gradings

Let $\Theta = \mathbb{Z}^r$ with standard basis $e_1, \ldots, e_r$. We partially order Θ by declaring that $I \geqslant J$ if $I_q \geqslant J_q$ for $1 \leqslant q \leqslant r$. The *diagonal* $\mathrm{diag}(\Theta)$ consists of the integer multiples of the vector $E = (1, \ldots, 1)$. By a *good* G*-algebra* we mean a finitely generated commutative k-algebra A on which G acts rationally by k-algebra automorphisms so that A has a good filtration as a G-module. We say that A is a *good* $G\Theta$*-algebra* if moreover A is Θ-graded by G-submodules,

$$A = \bigoplus_{I \in \Theta, I \geqslant 0} A_I$$

with

- $A_I A_J \subset A_{I+J}$;
- A is generated over A_0 by the A_{e_q}; and
- G acts trivially on A_0.

Motivated by the Segre embedding we define

$$\operatorname{diag}(A) = \bigoplus_{I \in \operatorname{diag}(\Theta)} A_I$$

and $\operatorname{Proj}(A) := \operatorname{Proj}(\operatorname{diag}(A))$. By an AG-module we will mean a Noetherian A-module M with compatible G-action. If moreover M is Θ-graded by G-submodules M_I so that $A_I M_J \subset M_{I+J}$, then we call M an *$AG\Theta$-module.*

Definition 3.1. Let A be a good G-algebra. We call an AG-module M *negligible* if M has finite good filtration dimension and each $H^i(\mathrm{SL}_N, M)$ is a Noetherian A^{SL_N}-module. Let $\mathcal{N}$ be the class of the negligible AG-modules.

Lemma 3.2. *$\mathcal{N}$ has the two-out-of-three property: If*

$$0 \to M' \to M \to M'' \to 0$$

is exact, and two of M', M, M'' are negligible, then so is the third.

Proof. The short exact sequence of Hochschild complexes [13, I.4.14]

$$0 \to C^*(\mathrm{SL}_N, M') \to C^*(\mathrm{SL}_N, M) \to C^*(\mathrm{SL}_N, M'') \to 0$$

is a bicomplex of A^{SL_N}-modules, so the long exact sequence

$$\cdots \to H^i(\mathrm{SL}_N, M') \to H^i(\mathrm{SL}_N, M) \to \cdots$$

is one of A^{SL_N}-modules, and A^{SL_N} is Noetherian by invariant theory. Also consider the long exact sequence

$$\cdots \to H^i(\mathrm{SL}_N, k[\mathrm{SL}_N/U] \otimes M') \to H^i(\mathrm{SL}_N, k[\mathrm{SL}_N/U] \otimes M) \to \cdots . \quad \square$$

More generally one has

Lemma 3.3. *Let $0 \to M_0 \to M_1 \to \cdots \to M_q \to 0$ be a complex of AG-modules whose homology modules $\ker(M_i \to M_{i+1})/\operatorname{im}(M_{i-1} \to M_i)$ are in $\mathcal{N}$ for $i = 0, \ldots, q$. If q of the M_i are in $\mathcal{N}$, so is the last one.*

Proof. This is a routine consequence of the two-out-of-three property. $\square$

4. Picard graded Cox rings

If V is a finite-dimensional k-vector space, we denote its dual by $V^{\#}$. For $1 \leqslant s \leqslant N$, let $\mathrm{Gr}(s)$ be the Grassmannian parametrizing s-dimensional subspaces of the dual $\nabla(\varpi_1)^{\#}$ of the defining representation of GL_N. Let $\mathcal{O}(1)$ denote as usual the ample generator of the Picard group of $\mathrm{Gr}(s)$. We wish to view it as a G-equivariant sheaf. To this end consider the parabolic subgroup $P = \{g \in G \mid g_{ij} = 0 \text{ for } i > N - s,\ j \leqslant N - s\}$ and identify $\mathrm{Gr}(s)$ with G/P. Then a G-equivariant vector bundle is the associated bundle of its fiber over P/P, where this fiber is a P-module. For the line bundle $\mathcal{O}(1)$ we let P act by the weight $\varpi_N - \varpi_{N-s}$ on the fiber over P/P. With this convention $\Gamma(\mathrm{Gr}(s), \mathcal{O}(1))$ is the Schur module $\nabla(\varpi_s)$, cf. [13, II, 2.16]. More generally, for $n \geqslant 0$ one has $\Gamma(\mathrm{Gr}(s), \mathcal{O}(n)) = \nabla(n\varpi_s)$. So

$$A\langle s\rangle = \bigoplus_{n\geqslant 0} \Gamma(\mathrm{Gr}(s), \mathcal{O}(n))$$

is a good $G\mathbb{Z}$-algebra. Recall that $\Theta = \mathbb{Z}^r$. Let $1 \leqslant s_i \leq N$ be given for $1 \leqslant i \leqslant r$. Then the Cox ring $A\langle s_1\rangle \otimes \cdots \otimes A\langle s_r\rangle$ of $\mathrm{Gr}(s_1) \times \cdots \times \mathrm{Gr}(s_r)$ is a good $G\Theta$-algebra. We put $C = C_0 \otimes A\langle s_1\rangle \otimes \cdots \otimes A\langle s_r\rangle$, where C_0 is a polynomial algebra on finitely many generators with trivial G-action and trivial grading. Then C is also a good $G\Theta$-algebra. We wish to prove

Proposition 4.1. *Every $CG\Theta$-module is negligible.*

The proof will be by induction on the rank r of Θ. It will be finished in Lemma 6.6. As base of the induction we use

Lemma 4.2. *A CG-module M that is Noetherian over C_0 is negligible.*

Proof. (Taken from [21].) As M is a finitely generated C_0-module it has only finitely many weights. Therefore the argument used in [7] to show that finite-dimensional G modules have finite good filtration dimension, applies to M.

As SL_N is reductive, it is well known [11, Theorem 16.9] that $H^0(\mathrm{SL}_N, M)$ is a finite $C_0^{\mathrm{SL}_N}$-module. So we argue by dimension shift. As M has only finitely many weights, one may choose s so large that all weights of $M \otimes k_{-(p^s-1)\rho}$ are antidominant, where $\rho = \sum_{i=1}^{N-1} \varpi_i$. Let St_s denote the sth Steinberg module $\mathrm{ind}_{B^+}^G(k_{-(p^s-1)\rho})$. Then $M \otimes \mathrm{St}_s = \mathrm{ind}_{B^+}^G(M \otimes k_{-(p^s-1)\rho})$ has by Kempf vanishing a good filtration and therefore $M \otimes \mathrm{St}_s \otimes \mathrm{St}_s$ has a good filtration [13, II, 4.21]. Then $H^i(\mathrm{SL}_N, M)$ is the cokernel of $H^{i-1}(\mathrm{SL}_N, M \otimes \mathrm{St}_s \otimes \mathrm{St}_s) \to H^{i-1}(\mathrm{SL}_N, M \otimes \mathrm{St}_s \otimes \mathrm{St}_s /M)$ for $i \geqslant 1$. $\square$

Notation 4.3. For $1 \leqslant q \leqslant r$ we denote by $C^{\widehat{q}}$ the subring $\bigoplus_{I_q=0} C_I$.

We further assume $r \geqslant 1$. The inductive hypothesis then gives:

Lemma 4.4. *Let $1 \leqslant q \leqslant r$. If the $CG\Theta$-module M is Noetherian over the subring $C^{\widehat{q}}$, then M is negligible.*

5. Coherent sheaves

We now have $\mathrm{Proj}(C) = \mathrm{Spec}(C_0) \times \mathrm{Gr}(s_1) \times \cdots \times \mathrm{Gr}(s_r)$. Call the projections of $\mathrm{Proj}(C)$ onto its respective factors $\pi_0, \ldots, \pi_r$. For $I \in \Theta$ define the coherent sheaf $\mathcal{O}(I) = \bigotimes_{i=1}^r \pi_i^*(\mathcal{O}(I_i))$. So $C = \bigoplus_{I\geqslant 0} \Gamma(\mathrm{Proj}(C), \mathcal{O}(I))$. For a $CG\Theta$-module M let $M^\sim$ be the coherent G-equivariant sheaf [5, 2.1], cf. [13, II, F.5], on $\mathrm{Proj}(C)$ constructed as in [12, II, 5.1] from the $\mathbb{Z}$-graded module $\mathrm{diag}(M) := \bigoplus_{I\in\mathrm{diag}(\Theta)} M_I$. Conversely, to a coherent sheaf $\mathcal{M}$ on $\mathrm{Proj}(C)$, we associate the Θ-graded C module

$$\Gamma_*(\mathcal{M}) = \bigoplus_{I\geqslant 0} \Gamma(\mathrm{Proj}(C), \mathcal{M}(I)),$$

where $\mathcal{M}(I) = M \otimes \mathcal{O}(I)$. We also put $H^t_*(\mathcal{M}) = \bigoplus_{I\geqslant 0} H^t(\mathrm{Proj}(C), \mathcal{M}(I))$. Recall from Section 3 that $E = (1, 1, ..., 1)$, so that $\mathcal{O}(E)$ is the natural very ample line bundle (relative to $\mathrm{Spec}(C_0)$) on the Segre product of the Grassmannians in Plücker embeddings.

Lemma 5.1. *If $\mathcal{M}$ is a G-equivariant coherent sheaf on* $\mathrm{Proj}(C)$*, then the $H^t_*(\mathcal{M})$ are $CG\Theta$-modules.*

Proof. So we have to show that $H^t_*(\mathcal{M})$ is Noetherian as a C-module. This is clear for $t > \dim(\mathrm{Proj}(C))$, so we argue by descending induction on t. Assume the result for all larger values of t. By Kempf vanishing $\bigoplus_{q\geqslant 0}\bigoplus_{n\geqslant 0} H^q(\mathrm{Gr}(s), \mathcal{O}(i+n))$ is a Noetherian $\bigoplus_{n\geqslant 0}\Gamma(\mathrm{Gr}(s), \mathcal{O}(n))$ module, for any $i \in \mathbb{Z}$, so by a Künneth theorem $\bigoplus_{q\geqslant 0} H^q_*(\mathrm{Proj}(C), \mathcal{O}(I))$ is a Noetherian C-module for any $I \in \Theta$. Now write $\mathcal{M}$ as a quotient of some $\mathcal{O}(iE)^a$ and use the long exact sequence

$$\cdots \to H^t_*(\mathcal{O}(iE)^a) \to H^t_*(\mathcal{M}) \to H^{t+1}(\dots) \to \cdots$$

to finish the induction step. $\square$

Notation 5.2. If M is a Θ-graded module and $I \in \Theta$, then $M(I)$ is the Θ-graded module with $M(I)_J = M_{I+J}$. Further $M_{\geqslant I}$ denotes $\bigoplus_{J\geqslant I} M_J$.

Lemma 5.3. *If $I \geqslant 0$, then the ideal $C_{\geqslant I}$ of C is generated by C_I. If M is a $CG\Theta$-module with $M_{nE} = 0$ for $n \gg 0$, then $M_{\geqslant nE} = 0$ for $n \gg 0$.*

Proof. The ideal is generated by C_I because C is generated over C_0 by the C_{e_i}. Let $m \in M_I$. Choose $J \geqslant 0$ with $I + J \in \mathrm{diag}(\Theta)$. Then mC_{J+qE} vanishes for $q \gg 0$, so $(mC)_{\geqslant I+J+qE} = 0$ for $q \gg 0$. Now use that M is finitely generated over C. $\square$

Lemma 5.4. *If M is a $CG\Theta$-module, then there is an n_0 so that if $I = nE = (n, \dots, n) \in \Theta$ with $n > n_0$, then $M_{\geqslant I} = \Gamma_*(M^\sim)_{\geqslant I}$.*

Proof. Recall [12, II, Ex. 5.9] that we have a natural map $\mathrm{diag}(M) \to \mathrm{diag}(\Gamma_*(M^\sim))$ whose kernel and cokernel live in finitely many degrees. Consider the maps $f : \mathrm{diag}(M) \otimes_{\mathrm{diag}(C)} C \to M$ and $g : \mathrm{diag}(M) \otimes_{\mathrm{diag}(C)} C \to \Gamma_*(M^\sim)$. If N is the kernel or cokernel of f or g then $N_{nE} = 0$ for $n \gg 0$. Now apply the previous lemma. $\square$

Lemma 5.5. *If M is a $CG\Theta$-module and $I \in \Theta$, then $M/M_{\geqslant I}$ is negligible.*

Proof. As M is finitely generated over C, there is $J < I$ with $M = M_{\geqslant J}$. Now note that for $1 \leqslant q \leqslant r$ and $K \in \Theta$ the module $M_{\geqslant K}/M_{\geqslant K+e_q}$ is negligible by 4.4. $\square$

Definition 5.6. In view of the above we call an equivariant coherent sheaf $\mathcal{M}$ on $\mathrm{Proj}(C)$ negligible when $\Gamma_*(\mathcal{M})$ is negligible.

The following lemma is now clear.

Lemma 5.7. *Let $I \in \Theta$. A G-equivariant coherent sheaf $\mathcal{M}$ on* $\mathrm{Proj}(C)$ *is negligible if and only if $\mathcal{M}(I)$ is negligible.*

Lemma 5.8. *Let*

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

be an exact sequence of G-equivariant coherent sheaves on $\mathrm{Proj}(C)$. *There is $I \in \Theta$ with*

$$0 \to \Gamma_*(\mathcal{M}')_{\geqslant I} \to \Gamma_*(\mathcal{M})_{\geqslant I} \to \Gamma_*(\mathcal{M}'')_{\geqslant I} \to 0$$

exact.

Proof. The line bundle $\mathcal{O}(E)$ is ample. Apply Lemma 5.3 to the homology sheaves of the complex

$$0 \to \Gamma_*(\mathcal{M}') \to \Gamma_*(\mathcal{M}) \to \Gamma_*(\mathcal{M}'') \to 0. \quad \square$$

Lemma 5.9. *For every $I \in \Theta$ the sheaf $\mathcal{O}(I)$ is negligible. If $\mathcal{F}$ is a G-equivariant coherent sheaf on* $\mathrm{Proj}(C)$ *so that $\Gamma_*(\mathcal{F})$ has finite good filtration dimension, then $\mathcal{F}$ is negligible.*

Proof. The first statement follows from the fact that C is negligible. As for the second, there is an equivariant exact sequence,

$$0 \to \mathcal{E} \to \mathcal{O}(i_q E) \otimes V_q \to \cdots \to \mathcal{O}(i_1 E) \otimes V_1 \to \mathcal{F} \to 0$$

with $\mathcal{E}$ a vector bundle, and each V_i a finite-dimensional G-module. Note that $\Gamma_*(\mathcal{E})_{\geqslant nE}$ has finite good filtration dimension for $n \gg 0$. Let

$$d = \lim_{n\to\infty} \dim_\nabla(\Gamma_*(\mathcal{E})_{\geqslant nE}).$$

If $d = 0$ then some $\Gamma_*(\mathcal{E})_{\geqslant J}$ has no higher SL_N-cohomology and is thus negligible by invariant theory [11, Theorem 16.9]. So we argue by induction on d. Say $d > 0$. As $\mathcal{E}$ is a vector bundle, there is short exact sequence of equivariant vector bundles $0 \to \mathcal{E} \to \mathcal{O}(nE) \otimes V \to \mathcal{E}' \to 0$, with V a finite-dimensional G-module. Any finite-dimensional G-module can be embedded into one with good filtration by [7], so we may assume V has good filtration. As $\mathcal{E}'$ has a smaller d [21, Lemma 2.1], induction applies. $\square$

6. Resolution of the diagonal

We write $X = \mathrm{Proj}(C)$, $Y = \mathrm{Proj}(C^{\hat{r}})$, $Z = \mathrm{Gr}(s)$, where $s = s_r$. So $X = Y \times Z$. We now recall the salient facts from [16], [18] about the functorial resolution of the diagonal in $Z \times Z$. As Z is the Grassmannian that parametrizes the s-dimensional subspaces of $\nabla(\varpi_1)^\#$, we have the tautological exact sequence of G-equivariant vector bundles on Z:

$$0 \to \mathcal{S} \to \nabla(\varpi_1)^\# \otimes \mathcal{O}_Z \to \mathcal{Q} \to 0,$$

where $\mathcal{S}$ has as fiber above a point the subspace V that the point parametrizes, and $\mathcal{Q}$ has as fiber above this same point the quotient $\nabla(\varpi_1)^\#/V$. Let π_1, π_2 be the respective projections $Z \times Z \to Z$. Then the composite of the natural maps

$\pi_1^*(\mathcal{S}) \to \nabla(\varpi_1)^\# \otimes \mathcal{O}_{Z\times Z}$ and $\nabla(\varpi_1)^\# \otimes \mathcal{O}_{Z\times Z} \to \pi_2^*(\mathcal{Q})$ defines a section of the vector bundle $\mathcal{H}om(\pi_1^*(\mathcal{S}), \pi_2^*(\mathcal{Q}))$ whose zero scheme is the diagonal $\operatorname{diag}(Z)$ in $Z\times Z$. Dually, we get an exact sequence $\mathcal{H}om(\pi_2^*(\mathcal{Q}), \pi_1^*(\mathcal{S})) \to \mathcal{O}_{Z\times Z} \to \mathcal{O}_{\operatorname{diag} Z} \to 0$, where $\mathcal{O}_{\operatorname{diag} Z}$ is the quotient by the ideal sheaf defining the diagonal. As the rank d of the vector bundle $\mathcal{E} = \mathcal{H}om(\pi_2^*(\mathcal{Q}), \pi_1^*(\mathcal{S}))$ equals the codimension of $\operatorname{diag}(Z)$ in $Z\times Z$, the Koszul complex

$$0 \to \bigwedge^d \mathcal{E} \to \cdots \to \mathcal{E} \to \mathcal{O}_{Z\times Z} \to \mathcal{O}_{\operatorname{diag} Z} \to 0$$

is exact. Now each $\bigwedge^i \mathcal{E}$ has a finite filtration whose associated graded is

$$\bigoplus S^\alpha \pi_1^*(\mathcal{S}) \otimes (S^{\tilde\alpha}\pi_2^*(\mathcal{Q}))^\#,$$

where α runs over partitions of i with at most $\operatorname{rank}(\mathcal{S})$ parts, so that moreover the conjugate partition $\tilde\alpha$ has at most $\operatorname{rank}(\mathcal{Q})$ parts.

Plan

Now the plan is this: Let $\pi_{1,2}$ be the projection of $Y\times Z\times Z$ onto the product $Y\times Z$ of the first two factors, let π_2 be the projection onto the middle factor Z, and so on. If M is a $CG\Theta$-module, tensor the pull-back along $\pi_{2,3}$ of the Koszul complex with $\pi_{1,3}^*(M^\sim)$, take a high Serre twist and then the direct image along $\pi_{1,2}$ to X. On the one hand $(\pi_{1,2})_*(\pi_{1,3}^*(M^\sim)\otimes \mathcal{O}_{\operatorname{diag} Z})$ is just $M^\sim$, but on the other hand the salient facts above allow us to express it in terms of negligible $CG\Theta$-modules. This will prove that M is negligible. We now proceed with the details.

Remark 6.1. Instead of functorially resolving the diagonal in $Z\times Z$, we could have functorially resolved the diagonal in $X\times X$.

Notation 6.2. On a product like $Y\times Z$ an exterior tensor product $\pi_1^*(\mathcal{F})\otimes\pi_2^*(\mathcal{M})$ is denoted $\mathcal{F}\boxtimes\mathcal{M}$.

Lemma 6.3. *Let $\mathcal{F}$ be a G-equivariant coherent sheaf on Y, and α a partition of i with at most s parts, $i\geqslant 0$. The sheaf $\mathcal{F}\boxtimes S^\alpha(\mathcal{S})$ on $X = Y\times Z$ is negligible.*

Proof. By the inductive assumption

$$\Gamma_*(\mathcal{F}) = \bigoplus_{I\in\mathbb{Z}^{r-1}, I\geqslant 0} \Gamma(Y, \mathcal{F}(I))$$

is a $C^{\hat r}$-module with finite good filtration dimension. The vector bundle $\mathcal{S}$ on $Z = G/P$ is associated with the irreducible P-representation with lowest weight $-\varepsilon_{N-s+1}$. This representation may be viewed as $\operatorname{ind}_{B^+}^P(-\varepsilon_{N-s+1})$, where $-\varepsilon_{N-s+1}$ also stands for the one-dimensional B^+ representation with weight $-\varepsilon_{N-s+1}$. Say $\rho : P \to P^-$ is the isomorphism which sends a matrix to its transpose inverse. Then $\operatorname{ind}_{B^+}^P(-\varepsilon_{N-s+1}) = \rho^* \operatorname{ind}_{B^-}^{P^-}(\varepsilon_{N-s+1})$. One finds that $S^\alpha(\mathcal{S})$ is associated with $\rho^*\operatorname{ind}_{B^-}^{P^-}(\sum_i \alpha_i\varepsilon_{N-s+i}) = \operatorname{ind}_{B^+}^P(-\sum_i \alpha_i\varepsilon_{N-s+i})$.

(This is the rule $S^\alpha(\nabla_{\mathrm{GL}_s}(\varpi_1)) = \nabla_{\mathrm{GL}_s}(\alpha)$ in disguise.) Then $S^\alpha(\mathcal{S})(n)$ is associated with $\mathrm{ind}_{B^+}^P(-\sum_i \alpha_i\varepsilon_{N-s+i} + n\varpi_N - n\varpi_{N-s})$. For $n \geqslant \alpha_1$ the weight $-\sum_i \alpha_i\varepsilon_{N-s+i} + n\varpi_N - n\varpi_{N-s}$ is an antidominant polynomial weight, so $\sum_{n\geqslant\alpha_1} \Gamma(Z, S^\alpha(\mathcal{S})(n))$ has a good filtration by transitivity of induction [13, I 3.5, 5.12]. Then $\Gamma_*(\mathcal{F}\boxtimes S^\alpha(\mathcal{S}))_{\geqslant I}$ has finite good filtration dimension [21, Lemma 2.1] for $I = (0, \dots, 0, \alpha_1)$ and the result follows from Lemma 5.9. $\square$

Assumption 6.4. Recall we are trying to prove that M is negligible. As in the proof of Lemma 5.9, we may reduce to the case that $M^\sim$ is a vector bundle. We further assume this.

Lemma 6.5. *For $n \gg 0$ the sheaf*

$$(\pi_{12})_*\Big(\pi_{13}^*(M^\sim) \otimes \big(\mathcal{O}(nE) \boxtimes \mathcal{O}(n)\big) \otimes \pi_{23}^*\Big(\bigwedge^i \mathcal{E}\Big)\Big)$$

is negligible.

Proof. The sheaf $\mathcal{O}(E) \boxtimes \mathcal{O}(1)$ is ample. So [12, Theorem 8.8] the sheaf in the lemma has a filtration with layers of the form

$$(\pi_{12})_*\big(\pi_{13}^*(M^\sim) \otimes \big(\mathcal{O}(nE) \boxtimes \mathcal{O}(n)\big) \otimes \pi_{23}^*\big(S^\alpha(\mathcal{S}) \boxtimes \mathcal{G}\big)\big).$$

Say $f : Y \times Z \to Y$ is the projection. Now use $(\pi_{12})_* \circ \pi_{13}^* = f^* \circ f_*$ and a projection formula for $(\pi_{12})_*$ to rewrite the layer in the form $(\mathcal{F} \boxtimes S^\alpha(\mathcal{S}))(I)$ for some $I \in \Theta$, with I depending on n. $\square$

End of proof of Proposition 4.1. Proposition 4.1 now follows from

Lemma 6.6. *$M^\sim$ is negligible.*

Proof. From the Koszul complex and the previous lemma we conclude [12, Theorem 8.8] that for $n \gg 0$ the sheaf

$$(\pi_{12})_*\big(\pi_{13}^*(M^\sim) \otimes \big(\mathcal{O}(nE) \boxtimes \mathcal{O}(n)\big) \otimes \pi_{23}^*(\mathcal{O}_{\mathrm{diag}(Z)})\big)$$

is negligible. This sheaf equals $M^\sim(I)$ for some $I \in \Theta$. $\square$

7. Differently graded Cox rings

Let $c : \{1, \dots, r\} \to \{1, \dots, q\}$ be surjective. Put $\Lambda = \mathbb{Z}^q$. We have a *contraction map*, also denoted c, from Θ to Λ with $c(I)_j = \sum_{i\in c^{-1}(j)} I_i$. Through this contraction we can view our Θ-graded C as Λ-graded. We now have the following generalization of Proposition 4.1.

Proposition 7.1. *Every $CG\Lambda$-module is negligible.*

This will be proved by descending induction on q, with fixed r. The case $q = r$ is clear. So let $q < r$ and assume the result for larger values of q. We may assume $c(r-1) = c(r) = q$. (Otherwise rearrange the factors.) Recall $X = \mathrm{Proj}(C)$, $X = Y \times Z$, with $Y = \mathrm{Proj}(C^{\widehat{r}})$, $Z = \mathrm{Proj}(A\langle s\rangle)$.

Notation 7.2. Let $\mathfrak{m}$ be the irrelevant maximal ideal $\bigoplus_{i>0} A\langle s\rangle_i$ of $A\langle s\rangle$. If M is a $CG\Lambda$-module, put $M_{\geqslant i} = \mathfrak{m}^i M$, and $\mathrm{gr}^i M = M_{\geqslant i}/M_{\geqslant i+1}$. If $I \in \Lambda$, put $(M_I)_{\geqslant i} = M_I \cap \mathfrak{m}^i M$, and $\mathrm{gr}^i M_I = (M_I)_{\geqslant i}/(M_I)_{\geqslant i+1}$. We put a $\mathbb{Z}^{q+1}$-grading on $\mathrm{gr}\, M = \bigoplus_i \mathrm{gr}^i M$ with

$$(\mathrm{gr}\, M)_I = \mathrm{gr}^{I_{q+1}} M_{(I_1,\dots,I_{q-1},I_q+I_{q+1})}.$$

In particular, all this applies when $M = C$. Then $\mathrm{gr}\, C$ may be identified with C and the $\mathbb{Z}^{q+1}$-grading on $\mathrm{gr}\, C$ is a contracted grading to which the inductive assumption applies. Write $\Phi = \mathbb{Z}^{q+1}$. Then $\mathrm{gr}\, M$ is a $CG\Phi$-module.

Let M be a $CG\Lambda$-module. By the inductive assumption $\mathrm{gr}\, M$ has finite good filtration dimension and each $H^i(\mathrm{SL}_N, \mathrm{gr}\, M)$ is a Noetherian $(\mathrm{gr}\, C)^{\mathrm{SL}_N}$-module. We still have to get rid of the grading. The filtration $M_{\geqslant 0} \supseteq M_{\geqslant 1} \cdots$ induces a filtration of the Hochschild complex [13, I,4.14] whence a spectral sequence

$$E(M) : E_1^{ij} = H^{i+j}(\mathrm{SL}_N, \mathrm{gr}^i M) \Rightarrow H^{i+j}(\mathrm{SL}_N, M).$$

It lives in two quadrants. The spectral sequence $E(M)$ is a direct sum of spectral sequences $E(M_I)$, $I \in \Lambda$. As each M_I has a finite filtration, each $E(M_I)$ stops, meaning that there is an a so that the differentials in $E_b^{**}(M_I)$ vanish for $b \geqslant a$. Thus $E_a^{**}(M_I) = E_\infty^{**}(M_I)$ is an associated graded of the abutment $H^*(\mathrm{SL}_N, M_I)$.

Lemma 7.3. *$E(M)$ also stops and its abutment is a Noetherian C^{SL_N}-module.*

Proof. The spectral sequence $E(C)$ is pleasantly boring: It does not just degenerate, even its abutment is the same as its E_1. The spectral sequence $E(M)$ is a module over it [3, Theorem 3.9.3], [15]. In particular, $E(M)$ is a module over C^{SL_N}. But $E_1^{**}(M)$ is Noetherian over $C^{\mathrm{SL}_N} = (\mathrm{gr}\, C)^{\mathrm{SL}_N}$. So the usual argument (see [22, Lemma 3.9] or [6, Lemma 7.4.4]) shows that $E(M)$ stops and that $E_\infty^{**}(M)$ is Noetherian over C^{SL_N}. As the filtrations on the abutments of the $E(M_I)$ are finite, it follows that the abutment of $E(M)$ is finitely generated over C^{SL_N}. □

Lemma 7.4. *M has finite good filtration dimension.*

Proof. As each M_I is finitely filtered, $\dim_\nabla(M_I) \leqslant \dim_\nabla(\mathrm{gr}\, M_I)$. □

This finishes the proof of Proposition 7.1.

8. Variations on the Grosshans grading

In this section we will be concerned with representations of SL_N. Mutatis mutandis everything also applies to other connected reductive groups. We now write $G = \mathrm{SL}_N$, with subgroups B^+, B^-, T, U defined in the usual manner. (So they are now the intersections with SL_N of the subgroups of GL_N that had these names.) As explained above, the Grosshans graded $\mathrm{gr}\, V$ of an SL_N-module V has a $\mathbb{Z}$-grading. We also need a Λ-graded version, where Λ is the weight lattice of SL_N. In [20] such a version was studied using a total order on weights known as the length–height order. It was claimed incorrectly in [21] that one might as well use the dominance order which is only a partial order. And it was claimed incorrectly in [21] that the resulting SL_N-module is isomorphic with $\operatorname{gr} V$. Both claims are correct when V has good filtration, but they are wrong in general. See Example 8.2 below. The claims are repeated in [22], [23]. Let us now introduce a Λ-graded version that is closer to the Grosshans graded than the version based on length–height order. (Length–height order was appropriate when dealing with the category of SL_N-modules as embedded into the larger category of B-modules.) Following Mathieu [17] we choose a second linear height function $E : \Lambda \otimes \mathbb{R} \to \mathbb{R}$ with $E(\alpha) > 0$ for every positive root α, but now with E injective on Λ. We define a total order on weights by first ordering them by Grosshans height, then for fixed Grosshans height by E. With this total order, denoted $\leqslant$, we put:

Definition 8.1. If V is a G-module, and λ is a weight, then $V_{\leqslant\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu \leqslant \lambda$ in the total order. For instance, $V_{\leqslant 0}$ is the module of invariants V^G. Similarly $V_{<\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu < \lambda$. Note that $V \mapsto V_{\leqslant\lambda}$ is a truncation functor for a saturated set of dominant weights [13, App. A]. So this functor fits in the usual highest weight category picture. As in [20], we form the Λ-graded module

$$\operatorname{gr}_\Lambda V = \bigoplus_{\lambda\in\Lambda} V_{\leqslant\lambda}/V_{<\lambda}.$$

Each $\operatorname{gr}_\lambda V = V_{\leqslant\lambda}/V_{<\lambda}$ has a B^+-socle $(\operatorname{gr}_\lambda V)^U = V_\lambda^U$ of weight λ. We always view V^U as a B^--module through restriction (inflation) along the homomorphism $B^- \to T$. Then $\operatorname{gr}_\lambda V$ embeds naturally in its 'good filtration hull' $\operatorname{hull}_\nabla(\operatorname{gr}_\lambda V) = \operatorname{ind}_{B^-}^G V_\lambda^U$. This good filtration hull has the same B^+-socle.

If λ is not dominant, then $\operatorname{gr}_\lambda V$ vanishes, because its socle vanishes. Note that $\bigoplus_{\operatorname{ht}(\lambda)=i} \operatorname{gr}_\lambda V$ is the associated graded of a filtration of $\operatorname{gr}_i V$, where $\operatorname{gr}_\lambda V$ refers to a graded component of $\operatorname{gr}_\Lambda V$ and $\operatorname{gr}_i V$ to one of $\operatorname{gr} V$. Both $\operatorname{gr}_\Lambda V$ and $\operatorname{gr} V$ embed into the good filtration hull $\operatorname{ind}_{B^-}^G V^U$, which is Λ-graded. But while $\operatorname{gr}_\Lambda V$ is a Λ-graded submodule of the hull, $\operatorname{gr} V$ need only be a $\mathbb{Z}$-graded submodule. Both $\operatorname{gr}_\Lambda V$ and $\operatorname{gr} V$ contain the socle of the hull.

Example 8.2. Take $p = 2$, $N = 3$. As a group we may take SL_3 or GL_3. Inside $\nabla(3\varpi_1 + \varpi_3) \oplus \nabla(3\varpi_2)$ take an indecomposable submodule V of codimension one. Then V has three composition factors. It has a one dimensional head and its socle is the direct sum of two irreducibles, whose highest weights have identical Grosshans height. It is easy to see that $\operatorname{gr}_\Lambda V$ has two indecomposable summands and $\operatorname{gr} V$ just one. And using the dominance order as suggested in [21] would not even lead to an associated graded of V. The head gets lost.

Although $\operatorname{gr}_\Lambda V$ need not coincide with $\operatorname{gr} V$ it shares some properties.

Lemma 8.3.

(1) *If A is a finitely generated k-algebra, so is* $\operatorname{gr}_\Lambda A$.
(2) *If A has good filtration, then* $\operatorname{gr}_\Lambda A$ *is isomorphic to* $\operatorname{gr} A$ *as k-algebra.*

Proof. Both $\operatorname{gr} A$ and $\operatorname{gr}_\Lambda A$ embed into their good filtration hull $\operatorname{ind}_{B^-}^G A^U$, notation $\operatorname{hull}_\nabla(\operatorname{gr} A)$, cf. [21, 2.2]. The argument of Mathieu (see proof of [21, Lemma 2.3]) that this $\operatorname{hull}_\nabla(\operatorname{gr} A)$ is the p-root closure of $\operatorname{gr} A$ applies just as well to the subalgebra $\mathrm{gr}_\Lambda A$. Indeed it would even apply to the subalgebra S of $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ generated by the socle of the hull. We argue as in the proof of [10, Theorem 9]. The finitely generated algebra $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ is integral over its finitely generated subalgebra S and $\mathrm{gr}_\Lambda A$ is an S-submodule of the hull. Then $\mathrm{gr}_\Lambda A$ must be finitely generated. When A has good filtration, $\mathrm{gr}_i A$ is already a direct sum of costandard modules. So then passing to the associated graded of the filtration of $\mathrm{gr}_i A$ makes no difference. And the algebra structure on both $\mathrm{gr}\, A$ and $\mathrm{gr}_\Lambda A$ agrees with the algebra structure on the hull by [23, Lemma 2.3]. $\square$

9. Proofs of the main results

Let us now turn to the proof of Theorem 1.1 for SL_N. Return to the notations introduced in Section 2. Thus $G = \mathrm{GL}_N$, with T its maximal torus. We assume the SL_N-algebra A has a good filtration and M is a Noetherian A-module on which SL_N acts compatibly. Put $\Lambda = \mathbb{Z}^{N-1}$ and identify Λ with a sublattice of $X(T)$ by sending $\lambda \in \Lambda$ to $\sum_i \lambda_i \varpi_i$. Also identify Λ with $X(T \cap \mathrm{SL}_N)$ through the restriction $X(T) \to X(T \cap \mathrm{SL}_N)$. Thus a dominant $\lambda \in \Lambda$ gets identified with a polynomial dominant weight. For such λ we may embed $\mathrm{gr}_\lambda A$ or $\mathrm{gr}_\lambda M$ into its good filtration hull which is a direct sum of restrictions to SL_N of the Schur module $\nabla_G(\lambda)$. On the Schur module $\nabla_G(\lambda)$ the center of G acts through λ. This makes it natural to use the Λ-grading on $\mathrm{gr}_\Lambda A$ and $\mathrm{gr}_\Lambda M$ to extend the action from SL_N to GL_N, making the center of GL_N act through λ on the graded pieces $\mathrm{gr}_\lambda A$ and $\mathrm{gr}_\lambda M$. We do that. Next we imitate Theorem 2.2 of [23].

Lemma 9.1. *Recall A has a good filtration, so that $\mathrm{gr}_\Lambda A = \mathrm{hull}_\nabla(\mathrm{gr}_\Lambda A)$. Let $R = \oplus_\lambda R_\lambda$ be a Λ-graded algebra with G-action such that $R_\lambda = (R_\lambda)_{\leqslant\lambda}$. Then every T-equivariant graded algebra homomorphism $R^U \to (\mathrm{gr}_\Lambda A)^U$ extends uniquely to a G-equivariant graded algebra homomorphism $R \to \mathrm{gr}_\Lambda A$.*

Proof. Use that $\mathrm{hull}_\nabla(\mathrm{gr}_\Lambda A)$ is an induced module. $\square$

As the algebra $(\mathrm{gr}_\Lambda A)^U = (\mathrm{gr}\, A)^U$ is finitely generated by Grosshans [10], it is also generated by finitely many weight vectors. Consider one such weight vector v, say of weight λ. Clearly λ is dominant. If $\lambda = 0$, map a polynomial ring $P_v := k[x]$ with trivial G-action to $\mathrm{gr}\, A$ by substituting v for x. Also put $D_v := 1$. Next assume $\lambda \neq 0$. Let $\ell = N - 1$ be the rank of Λ. Recall the Cox rings $A\langle i\rangle$ of Section 4. Define a T-action on the Λ-graded algebra

$$P = \bigotimes_{i=1}^{\ell} A\langle i\rangle$$

by letting T act on $\bigotimes_{i=1}^{\ell} \Gamma(\mathrm{Gr}(i), \mathcal{O}(m_i))$ through weight $\sum_i m_i \varpi_i$. So now we have a $(G \times T)$-action on P, and the T-action corresponds with the Λ-grading. Observe that by the tensor product property [13, Chap. G] the algebra P has a good filtration for the G-action. Let D be the scheme theoretic kernel of λ. So D has character group $X(D) = X(T)/\mathbb{Z}\lambda$ and $D = \mathrm{Diag}(X(T)/\mathbb{Z}\lambda)$ in the notations of [13, I,2.5]. The subalgebra $P^{1\times D}$ is a graded algebra with good filtration such that its subalgebra $P^{U\times D}$ contains a polynomial algebra on one generator x of weight $\lambda \times \lambda$. In fact, this polynomial subalgebra contains all the weight vectors in $P^{U\times D}$ whose weight is of the form $\nu \times \nu$. The other weight vectors in $P^{U\times D}$ have weight of the form $\mu \times \nu$ with ν an integer multiple of λ and $\mu < \nu$. These other weight vectors span an ideal in $P^{U\times D}$. By Lemma 9.1 one easily constructs a G-equivariant algebra homomorphism $P^{1\times D} \to \operatorname{gr}_\Lambda A$ that maps x to v. Write it as $P_v^{1\times D_v} \to \operatorname{gr}_\Lambda A$, to stress the dependence on v.

The direct product D of the D_v is a diagonalizable group. It acts on the tensor product C of the finitely many P_v. This C is Λ-graded. We have a graded algebra map $C^D \to \operatorname{gr}_\Lambda A$. It is surjective because its image has good filtration ([13, Chap. A]) and contains $(\operatorname{gr} A)^U$. We have proved

Lemma 9.2. *There is a graded G-equivariant surjection $C^D \to \operatorname{gr}_\Lambda A$, where the $(G \times D)$-algebra C is a good $G\Lambda$ algebra as in Proposition* 7.1*.*

Now recall M is a Noetherian A-module on which G acts compatibly, meaning that the structure map $A \otimes M \to M$ is a map of G-modules. Form the 'semidirect product ring' $A \ltimes M$ whose underlying G-module is $A \oplus M$, with product given by $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$. By Lemma 8.3 $\operatorname{gr}_\Lambda(A \ltimes M)$ is a finitely generated algebra, so we get

Lemma 9.3. $\operatorname{gr}_\Lambda M$ *is a Noetherian* $\operatorname{gr}_\Lambda A$*-module.*

This is of course very reminiscent of the proof of the lemma [11, Theorem 16.9] telling that M^G is a Noetherian module over the finitely generated k-algebra A^G. We will tacitly use its counterpart for diagonalizable actions, cf. [4], [13, I,2.11].

Now this lemma implies that $C \otimes_{C^D} \operatorname{gr}_\Lambda M$ is a $CG\Lambda$-module, so by Proposition 7.1 the following analogue of [23, Lemma 2.7] holds.

Lemma 9.4. *The module $C \otimes_{C^D} \operatorname{gr}_\Lambda M$ has finite good filtration dimension and each $H^i(\mathrm{SL}_N, C \otimes_{C^D} \operatorname{gr}_\Lambda M)$ is a Noetherian C^{SL_N}-module.*

Remark 9.5. Note that $C \otimes_{C^D} \operatorname{gr}_\Lambda M$ actually has finite Schur filtration dimension. Indeed we only need Proposition 7.1 for polynomial $CG\Lambda$-modules. On the other hand the reader may prefer to prove a version of Proposition 7.1 for SL_N rather than extending the action on $\operatorname{gr}_\Lambda A$ and $\operatorname{gr}_\Lambda M$ from SL_N to $G = \mathrm{GL}_N$. We now have to restrict back to SL_N anyway.

Now we get the analogue of [23, Lemma 2.8]

Lemma 9.6. *The module $\operatorname{gr}_\Lambda M$ has finite good filtration dimension and $\bigoplus_i H^i(\mathrm{SL}_N, \operatorname{gr}_\Lambda M)$ is a Noetherian A^{SL_N}-module.*

Proof. Extend the D-action on C to $C \otimes_{C^D} \operatorname{gr}_\Lambda M$ by using the trivial action on the second factor. Then we have a $(G \times D)$-module structure on $C \otimes_{C^D} \operatorname{gr}_\Lambda M$. As D is diagonalizable, C^D is a direct summand of C as a C^D-module [13, I,2.11] and $(C \otimes_{C^D} \operatorname{gr}_\Lambda M)^{1\times D} = \operatorname{gr}_\Lambda M$ is a direct summand of the G-module $C \otimes_{C^D} \operatorname{gr}_\Lambda M$. It follows that $\operatorname{gr}_\Lambda M$ also has finite good filtration dimension and it follows that each $H^i(\mathrm{SL}_N, C \otimes_{C^D} \operatorname{gr}_\Lambda M)^{1\times D} = H^i(\mathrm{SL}_N, \operatorname{gr}_\Lambda M)$ is a Noetherian $C^{\mathrm{SL}_N \times D}$-module. And there are only finitely many i for which $H^i(\mathrm{SL}_N, \operatorname{gr}_\Lambda M)$ is nonzero. But the action of $C^{\mathrm{SL}_N \times D}$ on $\operatorname{gr}_\Lambda M$ factors through $(\operatorname{gr}_\Lambda A)^{\mathrm{SL}_N}$, so we see that each $H^i(\mathrm{SL}_N, \mathrm{gr}_\Lambda M)$ is a Noetherian $(\mathrm{gr}_\Lambda A)^{\mathrm{SL}_N}$-module. And one always has $(\mathrm{gr}_\Lambda A)^{\mathrm{SL}_N} = (\mathrm{gr}_0 A)^{\mathrm{SL}_N} = A^{\mathrm{SL}_N}$. □

End of Proof of Theorems 1.1, 1.2. We see that each $\mathrm{gr}_\lambda M$ is negligible as an $(A^{\mathrm{SL}_N})G$-module. Enumerate the dominant weigths in Λ as λ_0, λ_1, ... according to our total order on weights. Note there are only finitely many dominant weights of given Grosshans height in Λ, so that the order type of the set of dominant weights in Λ is indeed just that of $\mathbb{N}$. (This would be false for the set of dominant weights in $X(T)$.) Using the two-out-of-three property specified in Lemma 3.2 we see by induction that $M_{\leqslant\lambda_n}$ is negligible as $(A^{\mathrm{SL}_N})G$-module. Moreover, as $\bigoplus_{i,\mu} H^i(\mathrm{SL}_N, \mathrm{gr}_\mu M)$ is Noetherian over A^{SL_N}, there are only finitely many nonzero $H^i(\mathrm{SL}_N, \mathrm{gr}_\mu M)$. So by a limit argument [13, I, Lemma 4.17] each $H^i(\mathrm{SL}_N, M)$ is a Noetherian A^{SL_N}-module. There is an m with $H^m(\mathrm{SL}_N, k[\mathrm{SL}_N/U] \otimes \mathrm{gr}_\lambda M) = 0$ for all $\lambda \in \Lambda$. So by a similar limit argument $H^m(\mathrm{SL}_N, k[\mathrm{SL}_N/U] \otimes M) = 0$ and M has finite good filtration dimension. This proves the theorem for the SL_N case. The GL_N case follows from the SL_N case, using that $H^i(\mathrm{GL}_N, M) = H^i(\mathrm{SL}_N, M)^{\mathbb{G}_m}$ for a GL_N-module M. Of course Theorem 1.2 follows from Theorem 1.1 by Proposition 2.2. □

Proof of Corollary 1.4. Now let A be any finitely generated commutative k-algebra on which SL_N acts rationally by k-algebra automorphisms. We argue as in the proof of [21, Prop. 3.8]. Recall again the following result of Mathieu [17], cf. [21, Lemma 2.3].

Lemma 9.7. *For every $x \in \mathrm{hull}_\nabla(\mathrm{gr}\, A)$, there is an integer $r \geqslant 0$, so that $x^{p^r} \in \mathrm{gr}\, A$.*

But $\mathrm{hull}_\nabla(\mathrm{gr}\, A)$ is finitely generated by Grosshans, so let us fix r so that for every $x \in \mathrm{hull}_\nabla(\mathrm{gr}\, A)$, one has $x^{p^r} \in \mathrm{gr}\, A$. By [8, Theorem 1.5, Remark 1.5.1] the ring $R = H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ is a finite module over the algebra

$$\bigotimes_{a=1}^{r} S^*((\mathfrak{gl}_n)^\#(2p^{a-1})) \otimes \mathrm{hull}_\nabla(\mathrm{gr}\, A).$$

This algebra has a good filtration by [2, 4.3], [13, Chap. G]. By Theorem 1.1 the ring R has finite good filtration dimension. Therefore there are only finitely many i with $E_2^{i*} \neq 0$ in the spectral sequence

$$E_2^{ij} = H^i(G/G_r, H^j(G_r, \mathrm{gr}\, A)) \Rightarrow H^{i+j}(G, \mathrm{gr}\, A).$$

So this spectral sequence stops, i.e. $E_s^{**} = E_\infty^{**}$ for some $s < \infty$. By the same theorem $H^*(G, R)$ is finite over the ring $H^0(G, \bigotimes_{a=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{a-1})) \otimes \mathrm{hull}_\nabla(\mathrm{gr}\, A))$, which is finitely generated by invariant theory [11, Theorem 16.9]. So $H^*(G, R) = E_2^{**}$ is a finitely generated k-algebra. Every page E_a^{**} is a differential graded algebra in characteristic p, so the pth power of an even element passes to the next page. Using this one sees that all pages are finitely generated as k-algebras. In particular, E_∞^{**} is finitely generated. As the spectral sequence lives in the first quadrant, the abutment is also finitely generated. □

Remark 9.8. Similarly the k-algebra $H^*(\mathrm{SL}_N, \mathrm{gr}_\Lambda A)$ is finitely generated. But $\mathrm{gr}_\Lambda A$ is even more graded than $\mathrm{gr}\, A$, and thus lies in the opposite direction of where we would like to go.

References

[1] K. Akin, D. Buchsbaum, J. Weyman, *Schur functors and Schur complexes*, Adv. in Math. **44** (1982), 207–278.

[2] H. H. Andersen, J.-C. Jantzen, *Cohomology of induced representations for algebraic groups*, Math. Ann. **269** (1984), 487–525.

[3] D. J. Benson, *Representations and Cohomology.* II. *Cohomology of Groups and Modules*, 2nd ed., Cambridge Studies in Advanced Mathematics, Vol. 31, Cambridge University Press, Cambridge, 1998.

[4] H. Borsari, W. Ferrer Santos, *Geometrically reductive Hopf algebras*, J. Algebra **152** (1992), 65–77.

[5] M. Brion, S. Kumar, *Frobenius Splitting Methods in Geometry and Representation Theory*, Progress in Mathematics, Vol. 231, Birkhäuser, Boston, 2005.

[6] L. Evens, *The Cohomology of Groups*, Oxford Mathematical Monographs, Oxford University Press, New York, 1991.

[7] E. M. Friedlander, B. J. Parshall, *Cohomology of Lie algebras and algebraic groups*, Amer. J. Math. **108** (1986), 235–253.

[8] E. M. Friedlander, A. A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), 209–270.

[9] J. A. Green, *Polynomial Representations of* GL_n, 2nd ed., Lecture Notes in Mathematics, Vol. 830, Springer, Berlin, 2007.

[10] F. D. Grosshans, *Contractions of the actions of reductive algebraic groups in arbitrary characteristic*, Invent. Math. **107** (1992), 127–133.

[11] F. D. Grosshans, *Algebraic Homogeneous Spaces and Invariant Theory*, Lecture Notes in Mathematics, Vol. 1673, Springer-Verlag, Berlin, 1997.

[12] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 52, Springer-Verlag, New York, 1977. Russian transl.: Р. Хартсхорн, *Алгебраическая геометрия*, Мир, М., 1981.

[13] J.-C. Jantzen, *Representations of Algebraic Groups*, Mathematical Surveys and Monographs, Vol. 107, Amer. Math. Soc., Providence, RI, 2003.

[14] G. Kempf, A. Ramanathan, *Multicones over Schubert varieties*, Invent. Math. **87** (1987), 353–363.

[15] W. S. Massey, *Products in exact couples*, Ann. of Math. **59** (1954), 558–569.

[16] M. Levine, V. Srinivas, J. Weyman, *K-Theory of twisted Grassmannians*, K-Theory **3** (1989), 99–121.

[17] O. Mathieu, *Filtrations of G-modules*, Ann. Sci. École Norm. Sup. **23** (1990), 625–644.

[18] P. Pragacz, V. Srinivas, V. Pati, *Diagonal subschemes and vector bundles*, available at `arXiv:math.AG/0609381`.

[19] A. Touzé, *Universal classes for algebraic groups*, available at `arXiv:0809.0989`.

[20] W. van der Kallen, *Lectures on Frobenius Splittings and B-modules*, Notes by S. P. Inamdar, Tata Institute of Fundamental Research, Bombay, and Springer-Verlag, Berlin, 1993.

[21] W. van der Kallen, *Cohomology with Grosshans graded coefficients*, in: *Invariant Theory in All Characteristics* (H. E. A. Eddy Campbell and D. L. Wehlau eds.), CRM Proceedings and Lecture Notes, Vol. 35, Amer. Math. Soc., Providence, RI, 2004, pp. 127-138.

[22] W. van der Kallen, *A reductive group with finitely generated cohomology algebras*, in: *Algebraic Groups and Homogeneous Spaces*, Mumbai 2004 (W. B. Mehta ed.), Tata Institute of Fundamental Research Studies in Mathematics, Vol. 19, Narosa, New Delhi, 2007.

[23] W. van der Kallen, *Finite good filtration dimension for modules over an algebra with good filtration*, J. Pure Appl. Algebra **206** (2006), 59–65.