Infinitesimal fixed points in modules with good filtration

Wilberd van der Kallen

January 1992

1 Introduction

Let $\mathfrak{g}$ be the Lie algebra of the connected semisimple algebraic group G defined over an algebraically closed field k of characteristic p, $p > 0$. If M is a G module with good filtration ([D]), we consider the module $M^{\mathfrak{g}}$ of fixed points under $\mathfrak{g}$ as a module for the image of G under the Frobenius homomorphism $G \to G$. This module is denoted $(M^{\mathfrak{g}})^{[-1]}$, cf. [J], and it has been conjectured that it also has a good filtration ([D]). We will give an example to show that this is too optimistic. In the other direction, we prove something stronger in rank 1. Namely, we show that if B is a Borel subgroup in SL_2 or PSL_2 and M is a B module with relative Schubert filtration (cf. [vdK]), then so is $(M^{\mathfrak{b}})^{[-1]}$.

Some preliminary work was done at the University of Virginia, where I much enjoyed the hospitality of Brian Parshall and Leonard Scott.

2 The rank 1 case

In this section let G be the group SL_2, or PSL_2, and B a Borel subgroup, with root α. Recall that G_r denotes the r-th Frobenius kernel of G and that $M^{\mathfrak{g}} = H^0(G_1, M)$ for any G module M ([J]).

Theorem 2.1 (Rank 1) *If M is a B module with relative Schubert filtration, then so is $(M^{\mathfrak{b}})^{[-1]}$. More generally, if $r \geq 1$, $s \geq 0$, and M is a B module with relative Schubert filtration, then so is $H^s(B_r, M)^{[-r]}$.*

Corollary 2.2 (Rank 1) *If M is a G module with good filtration, then so is $(M^{\mathfrak{g}})^{[-1]}$. More generally, if $r \geq 1$, $s \geq 0$, and M is a G module with good filtration, then so is $H^s(G_r, M)^{[-r]}$.*

Proof of corollary As explained in [D, p. 79], the second sentence follows from the first. Recall that if N is a module with relative Schubert filtration, then $\mathrm{ind}_B^G N$ has a good filtration (cf. [vdK, 2.25]). So the corollary follows from the Andersen–Jantzen formula ([J, I 6.12 and II 12.2]) $(\mathrm{ind}_B^G(N)^{\mathfrak{g}})^{[-1]} = \mathrm{ind}_B^G((N^{\mathfrak{b}})^{[-1]})$. □

2.3 Remark In the theorem one may not replace "relative Schubert" by "excellent".

2.4 Proof of the theorem Again the second sentence follows from the first. We may assume M is finite dimensional. Recall from [vdK] that $Q(\lambda) = k_\lambda$ if $(\alpha, \lambda) \leq 0$, $Q(\lambda) = \ker : \mathrm{ind}_B^G(-\lambda) \to k_{-\lambda}$ otherwise. Arguing by induction on dimension, what we have to prove is the claim that if

$$0 \to M' \to M \to Q(\lambda) \to 0$$

is exact and M' has relative Schubert filtration, then the map $M^{\mathfrak{b}} \to Q(\lambda)^{\mathfrak{b}}$ is either zero or surjective. This claim is obvious if the dimension of $Q(\lambda)^{\mathfrak{b}}$ is at most one. Assume it is larger. We have $\mathfrak{b} = \mathfrak{t} + \mathfrak{u}$ as usual, with $\mathfrak{u}$ generated by X_α. Inspection shows that $Q(\lambda)^{\mathfrak{b}}$ is just the image under X_α^{p-1} of $Q(\lambda)$. Because X_α^p acts trivially on any B module, one sees that $M^{\mathfrak{u}}$ contains the image under X_α^{p-1} of M and thus maps onto $Q(\lambda)^{\mathfrak{u}}$. The claim thus follows from exactness of taking fixed points under the action of $\mathfrak{t}$. □

3 Counterexample to the fixed point conjecture

We will give an example of a G module M with good filtration such that $(M^{\mathfrak{g}})^{[-1]}$ does not have a good filtration. By the above, no such examples exist when G has rank 1. Therefore we consider one of the next simplest cases. For the group we take $G = SL_3$ and for the characteristic we take 3. (Just to be specific). Say α and β are the simple roots of B (if one views roots of B as positive). For each parabolic P containing B we view the category of P modules as embedded in the category of B modules, so that we may always delete res_B^P from notations. We start with the exact sequence

$$\mathcal{E} : 0 \to k_\alpha \to \mathrm{ind}_B^{G_\alpha}(-\alpha) \to R \to 0$$

which defines R. Inside R we have its weight zero submodule R_0. Inspecting Pascal's triangle—which lists the structure constants of the B module $\mathrm{ind}_B^{G_\alpha}(\lambda)$ for any λ, cf. [J, II 5.2]—we see there is a B module map ϕ from $\mathrm{ind}_B^{G_\beta}(k_{-12\alpha-12\beta})$ onto $\mathrm{ind}_B^{G_\beta}(-\beta) \otimes R_0 \otimes k_{-12\alpha-7\beta}$. We tensor the extension $\mathcal{E}$ with $\mathrm{ind}_B^{G_\beta}(-\beta) \otimes k_{-12\alpha-7\beta}$, then pull it back along ϕ. The result is an extension

$$0 \to \mathrm{ind}_B^{G_\beta}(-\beta) \otimes k_{-11\alpha-7\beta} \to N \to \mathrm{ind}_B^{G_\beta}(-12\alpha - 12\beta) \to 0.$$

Here N has excellent filtration and we are going to take $M = \mathrm{ind}_B^G(N)$. By the Andersen–Jantzen formula ([J, I 6.12 and II 12.2])

$$(\mathrm{ind}_B^G(N)^{\mathfrak{g}})^{[-1]} = \mathrm{ind}_B^G((N^{\mathfrak{b}})^{[-1]})$$

we may study the fixed points under $\mathfrak{g}$ in M through those under $\mathfrak{b}$ in N. Now

$$(\mathrm{ind}_B^{G_\beta}(-12\alpha - 12\beta)^{\mathfrak{b}})^{[-1]} = \mathrm{ind}_B^{G_\beta}(-4\alpha - 4\beta)$$

and $(\mathrm{ind}_B^{G_\beta}(-\beta) \otimes k_{-12\alpha-7\beta} \otimes \mathrm{ind}_B^{G_\alpha}(-\alpha))^{\mathfrak{b}} = 0$, so $(N^{\mathfrak{b}})^{[-1]}$ is just the kernel of the surjective map

$$\mathrm{ind}_B^{G_\beta}(-4\alpha - 4\beta) \to \phi((\mathrm{ind}_B^{G_\beta}(-12\alpha - 12\beta))^{\mathfrak{b}}) = k_{-4\alpha-2\beta}.$$

Thus $\mathrm{ind}_B^G((N^{\mathfrak{b}})^{[-1]})$ is the kernel of the map $\mathrm{ind}_B^G(-4\alpha - 4\beta) \to \mathrm{ind}_B^G(-4\alpha - 2\beta)$. Further recall that the evaluation maps $\mathrm{ind}_B^G(-4\alpha - 4\beta) \to \mathrm{ind}_B^{G_\beta}(-4\alpha - 4\beta)$ and $\mathrm{ind}_B^G(-4\alpha - 2\beta) \to k_{-4\alpha-2\beta}$ are surjective. Thus

$$(\mathrm{ind}_B^G((N^{\mathfrak{b}})^{[-1]}))_{-4\alpha-4\beta} \neq 0, \text{ but } (\mathrm{ind}_B^G((N^{\mathfrak{b}})^{[-1]}))_{-4\alpha-2\beta} = 0,$$

so $(M^{\mathfrak{g}})^{[-1]}$ does not have a good filtration.

3.1 Remark To get examples with larger primes, tensor $\mathcal{E}$ with

$$\mathrm{ind}_B^{G_\beta}((-p/2+1/2)\beta) \otimes k_{(-p^2-p)\alpha+(-p^2+p/2+1/2)\beta}$$

and pull it back along $\phi : \mathrm{ind}_B^{G_\beta}(k_{(-p^2-p)(\alpha+\beta)}) \to \mathrm{ind}_B^{G_\beta}((-p/2+1/2)\beta) \otimes R_0 \otimes k_{(-p^2-p)\alpha+(-p^2+p/2+1/2)\beta}$. One may also get examples with $p = 2$ (replace p^2 by a sufficiently high power.) In fact, one may get examples for any prime number p and for any semisimple group of rank at least 2 (even for $PSL_2 \times PSL_2$.)

References

[D] S. Donkin, Good filtrations of rational modules for reductive groups, Proc. Symp. in Pure Math. 47, 69–80 (1987).

[J] J.C. Jantzen, Representations of algebraic groups, Academic Press, Orlando 1987.

[vdK] W. van der Kallen, Longest weight vectors and excellent filtrations, Math. Z. 201, 19–31 (1989).

Address

Wilberd van der Kallen, Mathematisch Instituut, Budapestlaan 6,
P. O. Box 80.010, 3508 TA Utrecht
The Netherlands