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Finite good filtration dimension for modules over an algebra with good filtration[☆]

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Dedicated to Eric M. Friedlander on his 60th birthday

Abstract

Let G be a connected reductive linear algebraic group over a field k of characteristic $p > 0$. Let p be large enough with respect to the root system. We show that if a finitely generated commutative k-algebra A with G-action has good filtration, then any noetherian A-module with compatible G-action has finite good filtration dimension.

MSC: 20G05; 20G10; 14L30

1. Introduction

Consider a connected reductive linear algebraic group G defined over a field k of positive characteristic p. We say that G has the cohomological finite generation property (CFG) if the following holds: Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. (So G acts on $\mathrm{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a k-algebra. Here, as in [8, I.4], we use the cohomology introduced by Hochschild, also known as 'rational cohomology'.

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In [13] we have shown that SL_2 over a field of positive characteristic has property (CFG), and in [14] we proved that SL_3 over a field of characteristic two has property (CFG). We conjecture that every reductive linear algebraic group has property (CFG). In this paper we show that this is at least a good heuristic principle: we derive one of the consequences of (CFG) for any simply connected semisimple linear algebraic group G that satisfies the following:

Hypothesis 1.1. Assume that for every fundamental weight ϖ_i the symmetric algebra $S^*(\nabla_G(\varpi_i))$ on the fundamental representation $\nabla_G(\varpi_i)$ has a good filtration.

Recall that this hypothesis is satisfied if $p \geqslant \max_i(\dim(\nabla_G(\varpi_i)))$, by [1, 4.1(5) and 4.3(1)]. This inequality is not necessary. For instance, SL_n satisfies the hypothesis for $n \leqslant 5$, by [13, Lemma 3.2]. When $p = 2$, the hypothesis does not hold for SL_n with $n \geqslant 6$, by [13, 3.3].

In the sequel let G be a connected reductive linear algebraic group over an algebraically closed field k of characteristic $p > 0$ with simply connected commutator subgroup for which Hypothesis 1.1 holds. Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. Let M be a noetherian A-module on which G acts compatibly. This means that the structure map $A \otimes M \to M$ is a G-module map. Our main result is

Theorem 1.2. *If A has good filtration, then M has finite good filtration dimension and each $H^i(G, M)$ is a noetherian A^G-module.*

When $A = k$ the theorem goes back to [4] and does not need Hypothesis 1.1. Unlike the proofs in [13] and [14], the proof of our theorem does not involve any cohomology of finite group schemes and is thus independent of the work of Friedlander and Suslin [5]. But without their work we would not have guessed the theorem. For clarity we will pull some material of [13] free from finite group schemes.

2. Recollections

Some unexplained notations, terminology, properties, ... can be found in [8]. We choose a Borel group $B^+ = TU^+$ and the opposite Borel group B^-. The roots of B^+ are positive. If $\lambda \in X(T)$ is dominant, then $\operatorname{ind}_{B^-}^G(\lambda)$ is the 'dual Weyl module' or 'costandard module' $\nabla_G(\lambda)$ with highest weight λ. The formula $\nabla_G(\lambda) = \operatorname{ind}_{B^-}^G(\lambda)$ just means that $\nabla_G(\lambda)$ is obtained from the Borel–Weil construction: $\nabla_G(\lambda)$ equals $H^0(G/B^-, \mathscr{L})$ for a certain line bundle on the flag variety G/B^-. In a *good filtration* $0 = V_{-1} \subseteq V_0 \subseteq V_1 \ldots$ of a G-module $V = \bigcup_i V_i$ the nonzero layers V_i/V_{i-1} are of the form $\nabla_G(\mu)$. As in [12] we will actually also allow a layer to be a direct sum of any number of copies of the same $\nabla_G(\mu)$, cf. [8, II.4.16 Remark 1]. This is much more convenient when working with infinite dimensional G-modules. It is shown in [3] that a module of countable dimension that has a good filtration in our sense also has a filtration that is a good filtration in the old sense. Note that the module M in our theorem has countable dimension. It would do little harm to restrict to modules of countable dimension throughout.

If V is a G-module, and $m \geqslant -1$ is an integer so that $H^{m+1}(G, \nabla_G(\mu) \otimes V) = 0$ for all dominant μ, then we say as in [4] that V has *good filtration dimension* at most m. The case $m = 0$ corresponds with V having a good filtration. And for $m \geqslant 0$ it means that V has a resolution

$$0 \to V \to N_0 \to \cdots \to N_m \to 0$$

in which the N_i have good filtration, in our sense. We say that V has good filtration dimension precisely m, notation $\dim_\nabla(V) = m$, if m is minimal so that V has good filtration dimension at most m. In that case $H^{i+1}(G, \nabla_G(\mu) \otimes V) = 0$ for all dominant μ and all $i \geqslant m$. In particular $H^{i+1}(G, V) = 0$ for $i \geqslant m$. If there is no finite m so that $\dim_\nabla(V) = m$, then we put $\dim_\nabla(V) = \infty$.

2.1. Filtrations

For simplicity assume also that G is semisimple. (until Remark 3.1.) If V is a G-module, and λ is a dominant weight, then $V_{\leqslant \lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu \leqslant \lambda$ in the dominance partial order [8, II.1.5]. For instance, $V_{\leqslant 0}$ is the module of invariants V^G. Similarly $V_{<\lambda}$ denotes the largest G-submodule all whose weights μ satisfy $\mu < \lambda$. As in [12], we form the $X(T)$-graded module

$$\mathrm{gr}_{X(T)} V = \bigoplus_{\lambda \in X(T)} V_{\leqslant \lambda}/V_{<\lambda}.$$

Each $V_{\leqslant \lambda}/V_{<\lambda}$, or $V_{\leqslant \lambda/<\lambda}$ for short, has a B^+-socle $(V_{\leqslant \lambda/<\lambda})^U = V_\lambda^U$ of weight λ. We always view V^U as a B^--module through restriction (inflation) along the homomorphism $B^- \to T$. Then $V_{\leqslant \lambda/<\lambda}$ embeds naturally in its 'good filtration hull' $\mathrm{hull}_\nabla(V_{\leqslant \lambda/<\lambda}) = \mathrm{ind}_{B^-}^G V_\lambda^U$. This good filtration hull has the same B^+-socle and by Polo it is the injective hull in the category $\mathscr{C}_\lambda$ of G-modules N that satisfy $N = N_{\leqslant \lambda}$. Compare [12, 3.1.10].

We convert the $X(T)$-graded module $\mathrm{gr}_{X(T)} V$ to a $\mathbb{Z}$-graded module through an additive height function $\mathrm{ht} : X(T) \to \mathbb{Z}$, defined by $\mathrm{ht} = 2\sum_{\alpha>0} \alpha^\vee$, the sum being over the positive roots. (Our ht is twice the one used by Grosshans [6], because we prefer to get even degrees rather than just integer degrees.) The Grosshans graded module is now

$$\mathrm{gr}\, V = \bigoplus_{i \geqslant 0} \mathrm{gr}_i\, V,$$

with

$$\mathrm{gr}_i\, V = \bigoplus_{\mathrm{ht}(\lambda)=i} V_{\leqslant \lambda/<\lambda}.$$

In other words, if one puts

$$V_{\leqslant i} := \sum_{\mathrm{ht}(\lambda) \leqslant i} V_{\leqslant \lambda},$$

then $\mathrm{gr}\, V$ is is the associated graded of the filtration $V_{\leqslant 0} \subseteq V_{\leqslant 1} \cdots$.

Let us apply the above to our finitely generated commutative k-algebra with G-action A. The Grosshans graded algebra $\operatorname{gr} A$ embeds in a good filtration hull, which Grosshans calls R, and which we call $\operatorname{hull}_\nabla(\operatorname{gr} A)$,

$$\operatorname{hull}_\nabla(\operatorname{gr} A) := \operatorname{ind}_{B^-}^G A^U = \bigoplus_i \bigoplus_{\operatorname{ht}(\lambda)=i} \operatorname{hull}_\nabla(A_{\leqslant\lambda}/A_{<\lambda}).$$

Grosshans [6] shows that A^U, $\operatorname{gr} A$, $\operatorname{hull}_\nabla(\operatorname{gr} A)$ are finitely generated k-algebras with $\operatorname{hull}_\nabla(\operatorname{gr} A)$ finite over $\operatorname{gr} A$. Mathieu studied $\operatorname{gr} A$ and $\operatorname{hull}_\nabla(\operatorname{gr} A)$ earlier in [10]. See also Popov [11].

Example 2.2. Consider the multicone [9]

$$k[G/U] := \operatorname{ind}_U^G k = \operatorname{ind}_{B^+}^G \operatorname{ind}_U^{B^+} k = \operatorname{ind}_{B^+}^G k[T] = \bigoplus_{\lambda \text{ dominant}} \nabla_G(\lambda).$$

It is its own Grosshans graded ring. Recall [9] that it is generated as a k-algebra by the finite dimensional sum of the $\nabla_G(\varpi_i)$, where ϖ_i denotes the ith fundamental weight.

Lemma 2.3. *Let A have a good filtration, so that* $\operatorname{gr} A = \operatorname{hull}_\nabla(\operatorname{gr} A)$. *Let $R = \oplus_i R_i$ be a graded algebra with G-action such that $R_i = (R_i)_{\leqslant i}$. Then every T-equivariant graded algebra homomorphism $R^U \to (\operatorname{gr} A)^U$ extends uniquely to a G-equivariant graded algebra homomorphism $R \to \operatorname{gr} A$.*

Proof. Use that $\operatorname{hull}_\nabla(\operatorname{gr} A)$ is an induced module. □

2.2. A graded polynomial $G \times D$-algebra with good filtration

We now extract a construction from [13]. It is hidden in the study of a Hochschild-Serre spectral sequence which in the present situation would correspond with the case where as normal subgroup one takes the trivial subgroup!

As the algebra $(\operatorname{gr} A)^U$ is finitely generated, it is also generated by finitely many weight vectors. Consider one such weight vector v, say of weight λ. Clearly λ is dominant. If $\lambda = 0$, map a polynomial ring $P_v := k[x]$ with trivial G-action to $\operatorname{gr} A$ by substituting v for x. Also put $D_v := 1$. Next assume $\lambda \neq 0$. Let ℓ be the rank of G. Define a T-action on the $X(T)$-graded algebra

$$P = \bigotimes_{i=1}^{\ell} S^*(\nabla_G(\varpi_i))$$

by letting T act on $\bigotimes_{i=1}^{\ell} S^{m_i}(\nabla_G(\varpi_i))$ through weight $\sum_i m_i \varpi_i$. So now we have a $G \times T$-action on P. Observe that by our key Hypothesis 1.1 and the tensor product property [8, Chapter G] the polynomial algebra P has a good filtration for the G-action. Let D be the scheme theoretic kernel of λ. So D has character group $X(D) = X(T)/\mathbb{Z}\lambda$ and

$D = \mathrm{Diag}(X(T)/\mathbb{Z}\lambda)$ in the notations of [8, I.2.5]. The subalgebra $P^{1\times D}$ is a graded algebra with good filtration such that its subalgebra $P^{U\times D}$ contains a polynomial algebra on one generator x of weight $\lambda \times \lambda$. In fact, this polynomial subalgebra contains all the weight vectors in $P^{U\times D}$ of weight $\mu \times \nu$ with $\mathrm{ht}(\mu) \geqslant \mathrm{ht}(\nu)$. The other weight vectors in $P^{U\times D}$ also have weight of the form $\mu \times \nu$ with ν a multiple of λ. These other weight vectors span an ideal in $P^{U\times D}$. Now assume A has a good filtration. By Lemma 2.3 one easily constructs a G-equivariant algebra homomorphism $P^{1\times D} \to \mathrm{gr}\, A$ that maps x to v. Write it as $P_v^{1\times D_v} \to \mathrm{gr}\, A$, to stress the dependence on v.

As new P we take the tensor product of the finitely many P_v and as diagonalized group D we take the direct product of the D_v. Then we have a graded algebra map $P^D \to \mathrm{gr}\, A$. It is surjective because its image has good filtration [8, Chapter A] and contains $(\mathrm{gr}\, A)^U$. The $G\times D$-algebra P is an example of what we called in [13] a graded polynomial $G\times D$-algebra with good filtration. We have proved

Lemma 2.5. *If A has a good filtration, then there is a graded polynomial $G \times D$-algebra P with good filtration and a graded G-equivariant surjection $P^D \to \mathrm{gr}\, A$.*

Now recall M is a noetherian A-module on which G acts compatibly, meaning that the structure map $A \otimes M \to M$ is a map of G-modules. Form the 'semi-direct product ring' $A \ltimes M$ whose underlying G-module is $A \oplus M$, with product given by $(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1)$. By Grosshans $\mathrm{gr}(A \ltimes M)$ is a finitely generated algebra, so we get

Lemma 2.6. gr M *is a noetherian* gr A*-module.*

This is of course very reminiscent of the proof of the lemma [7, Theorem 16.9] telling that M^G is a noetherian module over the finitely generated k-algebra A^G. We will tacitly use its counterpart for diagonalized actions, cf. [2,8, I.2.11].

Taking things together we learn that if A has a good filtration, then $P \otimes_{P^D} \mathrm{gr}\, M$ is what we called in [13] a finite graded P-module. Thus [13, Lemma 3.7] then tells us

Lemma 2.7. *Let A have good filtration. Then $P \otimes_{P^D} \mathrm{gr}\, M$ has finite good filtration dimension and each $H^i(G, P \otimes_{P^D} \mathrm{gr}\, M)$ is a noetherian P^G-module.*

Extend the D-action on P to $P \otimes_{P^D} \mathrm{gr}\, M$ by using the trivial action on the second factor. Then we have a $G \times D$-module structure on $P \otimes_{P^D} \mathrm{gr}\, M$. As D is diagonalized, P^D is a direct summand of P as a P^D-module [8, I.2.11] and $(P \otimes_{P^D} \mathrm{gr}\, M)^{1\times D} = \mathrm{gr}\, M$ is a direct summand of the G-module $P \otimes_{P^D} \mathrm{gr}\, M$. It follows that gr M also has finite good filtration dimension and it follows that each $H^i(G, P \otimes_{P^D} \mathrm{gr}\, M)^{1\times D} = H^i(G, \mathrm{gr}\, M)$ is a noetherian $P^{G\times D}$-module. But the action of $P^{G\times D}$ on gr M factors through $(\mathrm{gr}\, A)^G$, so we see that each $H^i(G, \mathrm{gr}\, M)$ is a noetherian $(\mathrm{gr}\, A)^G$-module. And one always has $(\mathrm{gr}\, A)^G = (\mathrm{gr}_0\, A)^G = A^G$. We conclude

Lemma 2.8. *Let A have good filtration. Then* gr M *has finite good filtration dimension and each $H^i(G, \mathrm{gr}\, M)$ is a noetherian A^G-module.*

3. Degrading

We still have to get rid of the grading. The filtration $M_{\leqslant 0} \subseteq M_{\leqslant 1} \cdots$ induces a filtration of the Hochschild complex [8, I.4.14] whence a spectral sequence

$$E(M) : E_1^{ij} = H^{i+j}(G, \mathrm{gr}_{-i} M) \Rightarrow H^{i+j}(G, M).$$

It lives in an unusual quadrant.

Assume that A has good filtration. Then by Lemma 2.8 $E_1(M)$ is a finitely generated A^G-module. So the spectral sequence lives in only finitely many bidegrees (i, j). Thus there is the same kind of convergence as one would have in a more common quadrant.

Choose A^G as ring of operators to act on the spectral sequence $E(M)$. As $E_1(M)$ is a noetherian A^G-module, it easily follows (even without the spectral sequence) that $H^*(G, M)$ is a noetherian A^G-module. To finish the proof of the theorem, we note that $A \otimes k[G/U]$ is also a finitely generated algebra with a good filtration and that $M \otimes k[G/U]$ is a noetherian module over it. So what we have just seen tells that $H^*(G, M \otimes k[G/U])$ is a noetherian $(A \otimes k[G/U])^G$-module. In particular, there is an $m \geqslant -1$ so that $H^{m+1}(G, M \otimes k[G/U]) = 0$.

Remark 3.1. Somewhere along the way we made the simplifying assumption that G is semisimple. So for the original G we have now proved that M has finite good filtration dimension with respect to the commutator subgroup H of G. But that is the same as having finite good filtration dimension with respect to G. Also, the fact that $H^i(H, M)$ is a noetherian A^H-module implies that $H^i(G, M)$ is a noetherian A^G-module by taking invariants under the diagonalizable center $Z(G)$.

Remark 3.2. We did not prove that M has a finite resolution by noetherian A-modules with compatible G-action and good filtration. We do not know how to start. One may embed M into the A-module $M \otimes k[G]$ with compatible G-action. It has good filtration, but it is not noetherian as an A-module.

Remark 3.3. The (CFG) property would imply that in Theorem 1.2 one does not need that A has good filtration, but only that it has finite good filtration dimension. It looks much harder to prove that version, even under Hypothesis 1.1.

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