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STEINBERG MODULES AND DONKIN PAIRS

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Abstract. First we prove that in characteristic $p > 0$ a module with good filtration for a group of type E_6 restricts to a module with good filtration for a group of type F_4. Thus we confirm a conjecture of Brundan for one more case. Our method relies on the canonical Frobenius splittings of Mathieu. Next we settle the remaining cases, with a computer-aided variation on the old method of Donkin.

1. Preliminaries

Our base field k is algebraically closed of characteristic p. Let G be a connected semisimple group and H a connected semisimple subgroup. (Good filtrations with more general groups are treated in [4].) We refer to [7] and [18] for unexplained terminology and notation.

Now choose a Borel subgroup B in G and a maximal torus T in B so that, if B^- is the opposite Borel subgroup, then $B \cap H$ and $B^- \cap H$ are Borel subgroups in H and $T \cap H$ is a maximal torus in H.

We follow the convention that the roots of B are positive. If $\lambda \in X(T)$ is dominant, then $\mathrm{ind}_B^G(-\lambda)$ is the dual Weyl module $\nabla_G(\lambda^*)$ with highest weight $\lambda^* = -w_0\lambda$ and lowest weight $-\lambda$. Its dual is the Weyl module $\Delta_G(\lambda)$. In a good filtration of a G-module the layers are of the form $\nabla_G(\mu)$.

Definition 1. We say that (G, H) is a Donkin pair if for any G-module M with good filtration, the H-module $\mathrm{res}_H^G M$ has good filtration.

Let X be a smooth projective B-variety with canonical bundle ω. (Generalizations to other varieties will be left to the reader.) There is by [12, §2] a natural map $\varepsilon : H^0(X, \omega^{1-p}) \to k$ so that $\phi \in H^0(X, \omega^{1-p})$ determines a Frobenius splitting if and only if $\varepsilon(\phi) = 1$. Let St_G be the Steinberg module of the simply connected cover $\tilde{G}$ of G. For simplicity of notation we further assume that St_G is actually a G-module. Its B-socle is the highest weight space $k_{(p-1)\rho}$.

Recall that a Frobenius splitting of X is called *canonical* if the corresponding ϕ is T-invariant and lies in the image of a B-module map $\mathrm{St}_G \otimes k_{(p-1)\rho} \to H^0(X, \omega^{1-p})$. If the group G needs to be emphasized, we will speak of a G-canonical splitting.

Remark 2. This agrees with the definition we used in [18, Definition 4.3.5]. Indeed if ϕ is a T-invariant in a B-module M, then ϕ lies in the image of a B-module map

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$\mathrm{St}_G \otimes k_{(p-1)\rho} \to M$ if and only if $X_\alpha^{(n)}\phi = 0$ for all simple roots α and all $n \geq p$. To see this, one must recall the presentation of Weyl modules. Thus let $\mathcal{U}(U)$ denote the hyperalgebra of the unipotent radical U of B. Then one needs the well-known lemma:

Lemma 3. *Let λ be dominant and let $v_{-\lambda^*}$ be a nonzero weight vector of lowest weight $-\lambda^*$ in $\Delta_G(\lambda)$. Then $v_{-\lambda^*}$ generates $\Delta_G(\lambda)$ as a $\mathcal{U}(U)$-module, and the annihilator of $v_{-\lambda^*}$ equals the left ideal of $\mathcal{U}(U)$ generated by the $X_\alpha^{(n)}$ with α simple and $n > (\lambda^*, \alpha^\vee)$.*

Proof. There are at least two known proofs. One is in the style of Verma modules. We will now give a reference for the proof that uses the geometry of Schubert varieties. Note that $\mathcal{U}(U)$ is a graded algebra graded by height. Therefore the left ideal in the lemma is the intersection of all ideals I of finite codimension that contain it and that lie inside the annihilator. But by the proof of [14, Proposition Fondamentale] such ideals I are equal to the annihilator. $\square$

Now suppose X is actually a G-variety.

Lemma 4. *X has a canonical splitting if and only if there is a G-module map ψ : $\mathrm{St}_G \otimes \mathrm{St}_G \to H^0(X, \omega^{1-p})$ so that $\varepsilon\psi \neq 0$.*

Proof. There is, up to scalar multiple, only one possibility for a map $\mathrm{St}_G \otimes \mathrm{St}_G \to k$. If $\varepsilon\psi \neq 0$, then the subspace of T-invariants in $\mathrm{St}_G \otimes k_{(p-1)\rho}$ maps isomorphically to k. Conversely, a map from $\mathrm{St}_G \otimes k_{(p-1)\rho}$ to a G-module M can be extended to $\mathrm{St}_G \otimes \mathrm{St}_G$ because the G-module generated by the image of $k_{(p-1)\rho}$ in $M \otimes \mathrm{St}_G^*$ is St_G. $\square$

We have the following fundamental result of Mathieu [10]. See also Remark 12 below.

Theorem 5. [11, 6.2] *Assume X has a canonical splitting and $\mathcal{L}$ is a G-linearized line bundle on X. Then $H^0(X, \mathcal{L})$ has a good filtration.*

2. Pairings

We will apply this to $X = G/B$. Of course the $\nabla_G(\mu)$ are of the form $H^0(X, \mathcal{L})$, see [7, I 5.12]. It follows that (G, H) is a Donkin pair if X has an H-canonical splitting. We also have a surjection $\mathrm{St}_G \otimes \mathrm{St}_G \to H^0(X, \omega^{1-p})$ by [7, II 14.20]. The composite with $H^0(X, \omega^{1-p}) \to k$ may be identified as in [8], [13] with the natural pairing on the self-dual representation St_G. All this suggests the following definition.

Definition 6. Assume there is an H-module map

$$\mathrm{St}_H^* \otimes \mathrm{St}_H \to \mathrm{St}_G^* \otimes \mathrm{St}_G$$

whose composite with the evaluation map $\mathrm{St}_G^* \otimes \mathrm{St}_G \to k$ is nonzero. Then we say (G, H) satisfies the pairing criterion.

Remark 7. Despite the notation, the $\tilde{H}$-module St_H need not be an H-module. Even if St_H is not an H-module, $\mathrm{St}_H^* \otimes \mathrm{St}_H$ is one. It may be better to replace $H \subset G$ with the homomorphism $\tilde{H} \to \tilde{G}$. Thus an operation like res_H^G would really mean restriction along $\tilde{H} \to \tilde{G}$.

Now we recall some old examples of Donkin pairs and show they satisfy the pairing criterion.

Example 8. Let G still be semisimple and connected. It is easy to see from the formulas in the proof of [8, 3.2] that the pairing criterion is satisfied for the diagonal G inside a product $G \times \cdots \times G$.

Example 9. Let H be the commutator subgroup of a Levi subgroup of a parabolic in the semisimple connected group G. Then, after passing to simply connected covers if necessary, St_H is a direct summand of $\mathrm{res}^G_H \mathrm{St}_G$, so again the pairing criterion is satisfied.

Lemma 10. *Let (G,H) satisfy the pairing criterion and let X be a smooth projective G-variety. If X has a G-canonical splitting, then it has an H-canonical one.*

Proof. Use Lemma 4. □

The following lemma was pointed out to me by Jesper Funch Thomsen.

Lemma 11. *Let X, Y be smooth projective G-varieties with canonical splitting. Then $X \times Y$ has a G-canonical splitting.*

Proof. Use Example 8. □

Remark 12. For the users of our book [18], let us now point out how to get Theorem 5. We have $G \times^B X = G/B \times X$ by Remark [18, 1.2.2], so [18, Lemma 4.4.2] applies to $Y = X$ in the notations of that lemma.

Remark 13. In Lemma 11 one cannot replace G with B. Here is an example. Take $G = SL_3$ in characteristic 2 and let Z be the Demazure resolution of a Schubert divisor. Then $H^0(Z, \omega_Z^{-1})$ is a nine-dimensional B-module. There is a fundamental representation V so that $H^0(Z, \omega_Z^{-1})$ is isomorphic to a codimension one submodule of the degree three part of the ring of regular functions on V. Using this, one checks with computer assisted computations that Z, $Z \times Z$, $Z \times Z \times Z$ have G-canonical splittings, while $Z \times Z \times Z \times Z$ does not.

Applying Theorem 5 to $X = G/B$ as indicated at the beginning of this section, we get the following representation theoretic criterion for Donkin pairs.

Theorem 14. (Pairing criterion) *Assume that (G,H) satisfies the pairing criterion. Then (G,H) is a Donkin pair.*

Remark 15. In fact the pairing criterion is satisfied if and only if G/B has an H-canonical splitting. Indeed suppose we are given a map $\mathrm{St}_H^* \otimes \mathrm{St}_H = \mathrm{St}_H \otimes \mathrm{St}_H \to H^0(G/B, \omega^{1-p})$ as in Lemma 4. We have to factor it through the surjection $\pi : \mathrm{St}_G^* \otimes \mathrm{St}_G \to H^0(G/B, \omega^{1-p})$. But the kernel K of π has a good filtration by [10] (or by the proof in [7, II 4.16]), so $\mathrm{Ext}^1_H(\mathrm{St}_H^* \otimes \mathrm{St}_H, \mathrm{res}^G_H K)$ vanishes by Theorem 5 and the main properties of good filtrations ([10, Theorem 1], [7, II 4.13]).

Our next aim is to treat the following example.

Example 16. For G we take the simply connected group of type E_6. From the symmetry of its Dynkin diagram we have a graph automorphism which is an involution. For H we take the group of fixed points of the involution. It is connected ([17, 8.2]) of type F_4. It has been conjectured by Brundan [3, 4.4] that (G,H) is a Donkin pair.

More generally, with our usual notations we have

Theorem 17. *Assume there are dominant weights σ_1, σ_2, σ_3, so that*

(1) *The highest weight $(p-1)\rho_G$ of St_G equals $\sigma_1+\sigma_2+\sigma_3$.*
(2) *$\sigma_1+\sigma_2$ and $\sigma_2+\sigma_3$ both restrict to the highest weight $(p-1)\rho_H$ of St_H.*
(3) *The natural map $\nabla_G(\sigma_1)\to\nabla_H(\mathrm{res}^B_{B\cap H}\sigma_1)$ is surjective.*

Then (G,H) is a Donkin pair. In fact it satisfies the pairing criterion.

Remark 18. If (G,H) is a Donkin pair and λ is dominant, then one knows that $\nabla_G(\lambda)\to\nabla_H(\mathrm{res}^B_{B\cap H}\lambda)=\mathrm{ind}^H_{H\cap B^-}(\mathrm{res}^{B^-}_{H\cap B^-}\lambda)$, induced by the projection of $\nabla_G(\lambda)$ onto its highest weight space, is surjective. (Exercise. Use a good filtration as in the proof of [7, II 4.16].)

Remark 19. Theorem 17 also applies to the Levi subgroup case of Example 9 (take $\sigma_1=0$). One hopes to find a more general method to attack at least all graph automorphisms. Theorem 17 applies if the graph automorphism is an involution and different simple roots in an orbit are perpendicular to each other. But for the graph automorphism of a group of type A_{2n} in characteristic $p>2$ there are no σ_1, σ_2, σ_3 as in the theorem. The coefficient of $\mathrm{res}^B_{B\cap H}\rho_G$ with respect to the fundamental weight that corresponds to the short root is four, which is too high.

Proof of Theorem 17. We will often write the restriction of a weight to $T\cap H$ with the same symbol as the weight. We will repeatedly use basic properties of Weyl modules and their duals. See [7, II 14.20] for surjectivity of cup product between dual Weyl modules and [7, II 2.13] for Weyl modules as universal highest weight modules. We first need a number of nonzero maps of H-modules. They are natural up to nonzero scalars that do not interest us.

The first map is

$$\varepsilon_H:\nabla_H(2(p-1)\rho_H)\to k$$

which detects Frobenius splittings on $H/(H\cap B)$. Together with the surjection

$$\nabla_H(\sigma_2)\otimes\nabla_H((p-1)\rho_G)\to\nabla_H(2(p-1)\rho_H)$$

this gives a nonzero map $\nabla_H(\sigma_2)\otimes\nabla_H((p-1)\rho_G)\to k$ and hence a nonzero

$$\eta_1:\nabla_H(\sigma_2)\to\nabla_H((p-1)\rho_G)^*.$$

The map $\nabla_G(\sigma_2+\sigma_3)\to\mathrm{St}_H$ is nonzero, hence surjective. The map $\nabla_G(\sigma_1)\to\nabla_H(\sigma_1)$ is surjective by assumption. In the commutative diagram

$$\begin{array}{ccc}\nabla_G(\sigma_2+\sigma_3)\otimes\nabla_G(\sigma_1) & \longrightarrow & \mathrm{St}_G\\ \downarrow & & \downarrow\\ \mathrm{St}_H\otimes\nabla_H(\sigma_1) & \longrightarrow & \nabla_H((p-1)\rho_G)\end{array}$$

the horizontal maps are also surjective. So the map

$$\eta_2:\nabla_H((p-1)\rho_G)^*\to\mathrm{St}_G^*$$

is injective. We obtain a nonzero

$$\eta_2\eta_1:\nabla_H(\sigma_2)\to\mathrm{St}_G^*.$$

The nonzero $\mathrm{St}_H \to \nabla_G(\sigma_2 + \sigma_3)$ combines with the map

$$\nabla_G(\sigma_1) \otimes \nabla_G(\sigma_2 + \sigma_3) \to \mathrm{St}_G$$

to yield

$$\nabla_G(\sigma_1) \otimes \mathrm{St}_H \to \mathrm{St}_G,$$

and combining this with $\eta_2\eta_1$ we get

$$\eta_3 : \nabla_H(\sigma_2) \otimes \nabla_G(\sigma_1) \otimes \mathrm{St}_H \to \mathrm{St}_G^* \otimes \mathrm{St}_G .$$

We claim that its image is detected by the evaluation map

$$\eta_4 : \mathrm{St}_G^* \otimes \mathrm{St}_G \to k.$$

This is because η_3 factors through $\nabla_H((p-1)\rho_G)^* \otimes \mathrm{St}_G$ on which the restriction of η_4 factors through $\nabla_H((p-1)\rho_G)^* \otimes \nabla_H((p-1)\rho_G)$, the map η_1 is nonzero, and the image of $\nabla_G(\sigma_1) \otimes \mathrm{St}_H \to \mathrm{St}_G$ maps onto $\nabla_H((p-1)\rho_G)$.

From the nontrivial $\eta_4\eta_3$ we get a nontrivial

$$\eta_5 : \nabla_H(\sigma_2) \otimes \nabla_G(\sigma_1) \to \mathrm{St}_H^* .$$

Then η_5 must be split surjective. Choose a left inverse

$$\eta_6 : \mathrm{St}_H^* \to \nabla_H(\sigma_2) \otimes \nabla_G(\sigma_1)$$

of η_5. This leads to

$$\eta_7 : \mathrm{St}_H^* \otimes \mathrm{St}_H \to \nabla_H(\sigma_2) \otimes \nabla_G(\sigma_1) \otimes \mathrm{St}_H$$

and the map we use in the pairing criterion is $\eta_3\eta_7$. Indeed the map $\mathrm{St}_H^* \to \mathrm{St}_H^*$ defined by $\eta_4\eta_3\eta_7$ equals $\eta_5\eta_6$, hence is nonzero. $\square$

3. The E_6–F_4 pair

We turn to the E_6–F_4 pair of Example 16. First observe that for $p > 13$ one could simply follow the method of [3] to prove that the pair is a Donkin pair. Indeed the restriction to F_4 of a fundamental representation then has its dominant weights in the bottom alcove. Looking a little closer and applying the linkage principle one can treat $p \geq 11$ in the same manner.

But for $p = 5$ one has $\varpi_4 \uparrow \varpi_1 + \varpi_4$ and for $p = 7$ one has $\varpi_1 \uparrow \varpi_1 + \varpi_4$. This makes it more trouble to see that the restriction of $\nabla_G(\varpi_4)$ has a good filtration with respective layers $\nabla_H(\varpi_1)$, $\nabla_H(\varpi_3)$, $\nabla_H(\varpi_3)$, $\nabla_H(\varpi_1 + \varpi_4)$, $\nabla_H(\varpi_2)$. For $p = 2$ or $p = 3$ it is even worse.

So let us apply Theorem 17 instead. We take $\sigma_1 = (p-1)(\varpi_1 + \varpi_3)$, $\sigma_2 = (p-1)(\varpi_2 + \varpi_4)$, $\sigma_3 = (p-1)(\varpi_5 + \varpi_6)$ in the notation of Bourbaki for E_6 [2, Planches]. Then $\mathrm{res}^B_{B\cap H}\, \varpi_i$ equals ϖ_4, ϖ_1, ϖ_3, ϖ_2, ϖ_3, ϖ_4 for $i = 1, \ldots, 6$, respectively.

First let $p = 2$. Then $\nabla_H(\operatorname{res}^B_{B\cap H}\sigma_1) = \nabla_H(\varpi_3 + \varpi_4)$ is irreducible. Indeed its dominant weights come in two parts. The weights 0, ϖ_4, ϖ_1, ϖ_3, $2\varpi_4$, $\varpi_1 + \varpi_4$, ϖ_2 lie in one orbit, and the highest weight lies in a different orbit under the affine Weyl group. To be more specific, $\varpi_1 - \rho_H \uparrow \varpi_3 + \varpi_4$, but $\varpi_4 - \rho_H \uparrow 0 \uparrow \varpi_4 \uparrow \varpi_1 \uparrow \varpi_3 \uparrow 2\varpi_4 \uparrow \varpi_1 + \varpi_4 \uparrow \varpi_2$. So $\nabla_G(\sigma_1) \to \nabla_H(\operatorname{res}^B_{B\cap H}\sigma_1)$ is surjective.

The case $p > 2$ remains. To see that $\nabla_G(\lambda) \to \nabla_H(\operatorname{res}^B_{B\cap H}\lambda)$ is surjective for $\lambda = \sigma_1$, it suffices to do this for $\lambda = \varpi_1$ and $\lambda = \varpi_3$. For $p > 3$ one could now use that $\nabla_H(\operatorname{res}^B_{B\cap H}\lambda)$ is irreducible for both $\lambda = \varpi_1$ and $\lambda = \varpi_3$, because each of the dominant weights of $\nabla_H(\operatorname{res}^B_{B\cap H}\lambda)$ is in a different orbit under the affine Weyl group.

But we need an argument that works for $p \geq 3$. Now $\nabla_G(\varpi_1)$ is a minuscule representation of dimension 27, and $\nabla_H(\varpi_4) = \nabla_H(\operatorname{res}^B_{B\cap H}\varpi_1)$ has dimension 26. There are 24 short roots and they have multiplicity one in $\nabla_H(\varpi_4)$. So the map from $M := \nabla_G(\varpi_1)$ to $\nabla_H(\varpi_4)$ hits at least 24 dimensions and its kernel consists of H-invariants. Indeed there are three weights of $\nabla_G(\varpi_1)$ that restrict to zero. In Bourbaki notation they are $\zeta_1 = 1/6(\varepsilon_8-\varepsilon_7-\varepsilon_6)+1/2(-\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5)$, $\zeta_2 = 1/6(\varepsilon_8-\varepsilon_7-\varepsilon_6)+1/2(\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5)$, $\zeta_3 = -1/3(\varepsilon_8-\varepsilon_7-\varepsilon_6)+\varepsilon_5$. Put $\zeta_4 = 1/6(\varepsilon_8-\varepsilon_7-\varepsilon_6)+1/2(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5)$, $\zeta_5 = 1/6(\varepsilon_8-\varepsilon_7-\varepsilon_6)+1/2(-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5)$. Then X_{α_1} induces an isomorphism $M_{\zeta_3} \to M_{\zeta_4}$ and it annihilates $M_{\zeta_1} + M_{\zeta_2}$. Similarly, X_{α_6} induces an isomorphism $M_{\zeta_2} \to M_{\zeta_4}$ and annihilates $M_{\zeta_1} + M_{\zeta_3}$. The same space is annihilated by X_{α_3}, which induces an isomorphism $M_{\zeta_2} \to M_{\zeta_5}$. Finally X_{α_5} induces an isomorphism $M_{\zeta_1} \to M_{\zeta_5}$ and annihilates $M_{\zeta_2} + M_{\zeta_3}$.

It follows that in $M_{\zeta_1} + M_{\zeta_2} + M_{\zeta_3}$ there is just a one-dimensional subspace of vectors annihilated by both $X_{\alpha_1} + X_{\alpha_6}$ and $X_{\alpha_3} + X_{\alpha_5}$. (These two operators come from the Lie algebra of H.) We conclude that $\operatorname{res}^G_H M$ has a good filtration and that $M \to \nabla_H(\varpi_4)$ is surjective. As $p > 2$, we then also have that $M \wedge M$ and $\operatorname{res}^G_H(M \wedge M)$ have a good filtration. It follows from the character that $M \wedge M = \nabla_G(\varpi_3)$. (We use the program `LiE` [19].) So $\operatorname{res}^G_H \nabla_G(\varpi_3)$ has a good filtration and therefore maps onto $\nabla_G(\operatorname{res}^B_{B\cap H}\varpi_3)$.

Summing up, we have shown

Theorem 20. *The* E_6–F_4 *pair is a Donkin pair. In fact it satisfies the pairing criterion.*

Remark 21. When Steve Donkin received this proof, he proceeded to show that one could also prove E_6–F_4 to be a Donkin pair with the 'ancient methods' of his book [4]. Of course he had to deal with more representations than we do. We will use his method in the last section to deal with the remaining cases of Brundan's conjecture, where we have no alternative yet.

4. Induction and canonical splitting

We finish the discussion of canonical splittings with an analogue of Proposition [11, 5.5]. This makes a principle from [10] more explicit. The result was explained to us by O. Mathieu at a reception of the mayor of Aarhus in August 1998. It shows once more that canonical splittings combine well with Demazure desingularisation of Schubert varieties.

Proposition 22. *Let X be a projective B-variety with canonical splitting. Let P be a minimal parabolic. Then $P \times^B X$ has a canonical splitting.*

Corollary 23. *The same conclusion holds for any parabolic subgroup.*

Proof. If P is not minimal, take a Demazure resolution $Z = P_1 \times^B P_2 \times^B \cdots \times^B P_r/B$ of P/B and apply the proposition to get a canonical splitting on $P_1 \times^B P_2 \times^B \cdots \times^B P_r \times^B X$. Then push the splitting forward ([12, Prop. 4]) to $P \times^B X$. $\square$

Proof of Proposition. We use notation as in [18, Ch. 4, A.4]. Let ζ be the highest weight of St and s the simple reflection corresponding to P. One checks as in [18, A.4.6] that

$$\mathcal{E}nd_F(P \times^B X) = (P \times^B \mathcal{E}nd_F(X)) \otimes \pi^*\mathcal{L}(s\zeta - \zeta),$$

where $\pi : P \times^B X \to P/B$. We are given a map $\phi : k_\zeta \otimes \mathrm{St} \to \mathrm{End}_F(X)$. The required map $\psi : k_\zeta \otimes \mathrm{St} \to \mathrm{End}_F(P \times^B X)$ may be constructed by composing maps

$$\begin{array}{rcl} k_\zeta \otimes \mathrm{St} & \cong & k_{-s\zeta} \otimes \mathrm{ind}_B^P(k_{\zeta+s\zeta} \otimes \mathrm{St}) \\ & \to & k_{-s\zeta} \otimes \mathrm{ind}_B^P(k_{s\zeta} \otimes \mathrm{End}_F(X)) \\ & \cong & k_{-s\zeta} \otimes H^0(P \times^B X, P \times^B (\mathcal{E}nd_F(X)[s\zeta])) \\ & \cong & \mathrm{End}_F(P \times^B X, B \times^B X) \\ & \to & \mathrm{End}_F(P \times^B X). \end{array}$$

Here $k_{-s\zeta}$ is identified with the weight space of weight $-s\zeta$ of

$$H^0(P \times^B X, \pi^*\mathcal{L}(-\zeta)).$$

An element of that weight space has divisor $(p-1)B \times^B X = (p-1)X$.

To see that the image of ψ is not in the kernel of

$$\varepsilon_{P \times^B X} : \mathrm{End}_F(P \times^B X) \to k,$$

it suffices to show that the diagram

$$\begin{array}{ccccc} k_\zeta \otimes \mathrm{St} & \to & \mathrm{End}_F(P \times^B X, B \times^B X) & \to & \mathrm{End}_F(P \times^B X) \\ \| & & \downarrow & & \downarrow \\ k_\zeta \otimes \mathrm{St} & \xrightarrow{\phi} & \mathrm{End}_F(X) & \to & k \end{array}$$

commutes. Now

$$\begin{array}{ccc} k_{-s\zeta} \otimes \mathrm{ind}_B^P(k_{\zeta+s\zeta} \otimes \mathrm{St}) & \longrightarrow & k_\zeta \otimes \mathrm{St} \\ \downarrow & & \downarrow \\ k_{-s\zeta} \otimes \mathrm{ind}_B^P(k_{s\zeta} \otimes \mathrm{End}_F(X)) & \longrightarrow & \mathrm{End}_F(X) \end{array}$$

commutes and by restricting to the trivial fibration $BsB \times^B X \to BsB/B$ one shows using the following lemma that the bottom map in this last diagram agrees with the map that factors through $\mathrm{End}_F(P \times^B X, B \times^B X)$. $\square$

Lemma 24. *Let A be a commutative k-algebra. Then*

$$\mathrm{End}_F(A[t]) = \mathrm{End}_F(A) \otimes \mathrm{End}_F(k[t]) = \mathrm{End}_F(A) \otimes k[t]$$

and the map $\mathrm{End}_F(A[t],(t)) \to \mathrm{End}_F(A)$ is induced by the map

$$t^{p-1}k[t] = t^{p-1} * \mathrm{End}_F(k[t]) = \mathrm{End}_F(k[t],(t)) \to \mathrm{End}_F(k) = k$$

which sends $t^{p-1}f(t)$ to $f(0)$.

Proof. Straightforward, provided one keeps in mind how $\mathrm{End}_F(R)$ is an R-module ([18, 4.3.3]). Compare also [18, A.4.5]. $\square$

5. More Donkin pairs

In this section we do not use the pairing criterion. Instead we return to the methods of Donkin's book [4], combined with computer calculations of characters, of linkage, and of the Jantzen sum formula.

Let G, H be as before, with G simply connected. In fact H will be the commutator subgroup of the group of fixed points of an involution of G which leaves invariant the maximal torus T and the Borel subgroup B. We refer to [16] for the classification of the possibilities, assuming $p > 2$. (Of course involutions of the simply connected G are lifted [17, 9.16] from the involutions of the corresponding adjoint group, which are treated in [16].)

Remark 25. Let H be the fixed point group of an involution that leaves T and B invariant in the simply connected semisimple G. Then H is connected reductive by [17, 8.2]. Now an H-module has good filtration in the sense of [4] if and only if its restriction to the commutator subgroup of H has good filtration. That is why we look only at semisimple subgroups H.

Let $\mathcal{M}$ denote the set of finite dimensional G-modules M with good filtration for which $\operatorname{res}^G_H M$ has good filtration. Let $\mathcal{S}$ denote the set of dominant weights λ of G so that $\nabla_G(\lambda) \in \mathcal{M}$. As always we try to show that all dominant weights of G are in $\mathcal{S}$. For this purpose we recall some useful lemmas.

Lemma 26. (1) *If $M_1 \oplus M_2 \in \mathcal{M}$, then $M_1 \in \mathcal{M}$.*
(2) *If $0 \to M' \to M \to M'' \to 0$ is exact, and $M' \in \mathcal{M}$, then $M \in \mathcal{M}$ if and only if $M'' \in \mathcal{M}$.*
(3) *If M_1, $M_2 \in \mathcal{M}$, then $M_1 \otimes M_2 \in \mathcal{M}$.*

If M is a G-module with good filtration, write $\operatorname{supp}_\nabla(M)$ for the set of dominant weights λ so that $\nabla_G(\lambda)$ occurs as a layer in a good filtration of M. We order the dominant weights of G by the partial order in which $\mu \leq \lambda$ if and only if $\lambda - \mu$ is in the closed cone spanned by the positive roots. In particular, if λ is dominant, then $0 \leq \lambda$, and all dominant weights μ of $\nabla_G(\lambda)$ satisfy $\mu \leq \lambda$. We say that a filtration of M is a good filtration *adapted* to the partial order if there are λ_i so that the i-th layer is a direct sum of copies of $\nabla_G(\lambda_i)$, and $i \leq j$ if $\lambda_i \leq \lambda_j$. (So we still call it a good filtration, even though $\nabla_G(\lambda_i)$ may have multiplicity in the i-th layer.) If M has a good filtration, then it also has one adapted to the partial order, by the proof of [7, II 4.16].

Lemma 27. *Let $M \in \mathcal{M}$ and $\lambda \in \operatorname{supp}_\nabla(M)$. Assume for every weight μ in $\operatorname{supp}_\nabla(M)$, distinct from λ, that one of the following holds:*

(1) *$\mu < \lambda$ and $\mu \in \mathcal{S}$.*
(2) *μ and λ are in different orbits under the affine Weyl group.*

Then $\lambda \in \mathcal{S}$.

Proof. We may first replace M by an indecomposable direct summand M_1 with $\lambda \in \operatorname{supp}_\nabla(M_1)$. The linkage principle says that we thus get rid of the second possibility in the lemma. Then in a good filtration adapted to the partial order, the module $\nabla_G(\lambda)$ occurs only as a summand of the top layer, which is in $\mathcal{M}$ by Lemma 26. □

Lemma 28. *Let λ be a dominant weight of G. If λ is in the bottom alcove, or if the Jantzen sum formula yields zero, then $\nabla_G(\lambda)$ is irreducible.*

Proof. See [7, II Cor. 5.6 and 8.21] □

1. The pairs E_8, D_8 and E_8, E_7A_1

Say G is of type E_8 in characteristic $p > 2$ and H is the fixed point group of an involution. There are two cases, up to conjugacy. One may have H of type D_8 or one may have H of type E_7A_1.

In either case we wish to show that G, H is a Donkin pair. In other words, we want all dominant weights to be in $\mathcal{S}$. We will argue by induction along the partial order. Thus when trying to prove that $\lambda \in \mathcal{S}$, we shall always assume that $\mu \in \mathcal{S}$ for $\mu < \lambda$. Of course the zero weight is in $\mathcal{S}$, so say λ is nonzero. If λ is not a fundamental weight, write $\lambda = \lambda_1 + \lambda_2$ where λ_i are nonzero dominant weights. As $\lambda_i < \lambda$, we may apply Lemma 27 with $M = \nabla_G(\lambda_1) \otimes \nabla_G(\lambda_2)$ to conclude that $\lambda \in \mathcal{S}$.

The fundamental weights remain. Observe that $\varpi_8 < \varpi_1 < \varpi_7 < \varpi_2 < \varpi_6 < \varpi_3 < \varpi_5 < \varpi_4$. But we will not discuss them in this exact order.

To see that $\varpi_8 \in \mathcal{S}$ we compute the character of $\operatorname{res}^G_H \nabla_G(\varpi_8)$ with the program `LiE` [19], decompose this character in terms of Weyl characters, and use Lemma 28 to see that $\operatorname{res}^G_H \nabla_G(\varpi_8)$ has a composition series whose factors are irreducible (dual) Weyl modules. Here we use a Java applet of Lauritzen for the Jantzen sum formula.

To see that $\varpi_1 \in \mathcal{S}$ we may argue the same way if H is of type D_8. If H is of type E_7A_1, let K be the subgroup of type E_7 in H, and F the subgroup of type A_1. Then G, K is a Donkin pair (Levi subgroup case), so we may consider a good filtration adapted to the partial order (on weights for K) of $\operatorname{res}^G_K \nabla_G(\varpi_1)$. This is a filtration by H-modules. It suffices to show that its layers have good filtration as H-modules. Let N be such a layer and let λ_1 be the dominant weight of K so that $\operatorname{res}^H_K N$ is a direct sum of copies of $\nabla_K(\lambda_1)$. We may observe as in [5, 1.3(1)] that N is isomorphic with the exterior tensor product of the K-module $\nabla_K(\lambda_1)$ and the F-module $\operatorname{Hom}_K(\nabla_K(\lambda_1), N)$. The character of N is the character of some $\nabla_H(\lambda_1, \lambda_2)$ where λ_2 is a dominant weight for F. (From now on we do not mention the computer calculations that are needed to support such statements.) Moreover, $\nabla_F(\lambda_2)$ is irreducible, so that the natural map $N \to \nabla_H(\lambda_1, \lambda_2)$ is an isomorphism, and thus $\varpi_1 \in \mathcal{S}$.

To see that $\varpi_2 \in \mathcal{S}$, we apply Lemma 27 with $M = \nabla_G(\varpi_1) \otimes \nabla_G(\varpi_8)$. (Note ϖ_1, $\varpi_8 < \varpi_2$, so that indeed $M \in \mathcal{M}$ by the inductive assumption.) One may find the necessary statement about linkage in Donkin's book. (This is no accident, as we follow him in our choices.) We also checked the nonlinkage with a straightforward Mathematica program.

Observe that if $M \in \mathcal{M}$ and $s < p$, then $\wedge^s M \in \mathcal{M}$ because it is a direct summand of $M^{\otimes s}$.

To see that $\varpi_7 \in \mathcal{S}$, we similarly apply Lemma 27 with $M = \nabla_G(\varpi_8) \wedge \nabla_G(\varpi_8)$. (Recall $p > 2$.) To get $\varpi_3 \in \mathcal{S}$, use $M = \nabla_G(\varpi_1) \wedge \nabla_G(\varpi_1)$. To get $\varpi_4 \in \mathcal{S}$, use $M = \nabla_G(\varpi_2) \wedge \nabla_G(\varpi_2)$. To get $\varpi_5 \in \mathcal{S}$, use $M = \nabla_G(\varpi_1) \otimes \nabla_G(\varpi_2)$ if $p = 3$, and $M = \wedge^4 \nabla_G(\varpi_8)$ if $p > 3$. To get $\varpi_6 \in \mathcal{S}$, use $M = \nabla_G(\varpi_1) \otimes \nabla_G(\varpi_1)$ if $p = 3$, and $M = \wedge^3 \nabla_G(\varpi_8)$ if $p > 3$.

2. The pair E_6, A_5A_1

Let G be the simply connected group of type E_6 in characteristic $p > 2$ and let H be the fixed point group of an inner involution such that H is of type A_5A_1 and the involution commutes with the graph automorphism. We wish to show again this is a Donkin pair. We argue as in the E_8, E_7A_1 case.

If $\lambda = \varpi_1$ or ϖ_2, then we argue with exterior tensor products as we did to show $\varpi_1 \in \mathcal{S}$ for the E_8, E_7A_1 pair.

To treat ϖ_3 we use $M = \nabla_G(\varpi_1) \wedge \nabla_G(\varpi_1)$. To get $\varpi_4 \in \mathcal{S}$, use $M = \nabla_G(\varpi_2) \wedge \nabla_G(\varpi_2)$. The remaining two fundamental weights are in $\mathcal{S}$ by symmetry.

3. The pair E_6, C_4

Let G be the simply connected group of type E_6 in characteristic $p > 2$ and let H be the fixed point group of an outer involution such that H is of type C_4 and the involution commutes with the graph automorphism.

The module $\nabla_G(\varpi_2)$ is the Lie algebra of the adjoint form of G. Its restriction $\operatorname{res}^G_H \nabla_G(\varpi_2)$ has a six-dimensional weight space for the weight zero, just like $\nabla_G(\varpi_2)$. We claim that it contains no nonzero invariant. Indeed we may choose the involution so that $X_{\alpha_3} + X_{\alpha_5}$, $X_{\alpha_1} + X_{\alpha_6}$, $X_{\alpha_3+\alpha_4} + X_{\alpha_5+\alpha_4}$, X_{α_2} are in the Lie algebra of H, where we have put $X_{\alpha_3+\alpha_4} = [X_{\alpha_3}, X_{\alpha_4}]$ and $X_{\alpha_5+\alpha_4} = [X_{\alpha_4}, X_{\alpha_5}]$. The only element in the weight zero weight space of $\nabla_G(\varpi_2)$ that is annihilated by all these elements is the zero vector. Now $\operatorname{res}^G_H \nabla_G(\varpi_2)$ contains an irreducible $\nabla_H(2\varpi_1)$ and the quotient by that submodule is either irreducible, or $p = 3$ and there are two composition factors, one of which is one-dimensional. (This also uses the Jantzen sum formula.) As there is no invariant in $\operatorname{res}^G_H \nabla_G(\varpi_2)$ and there is no extension between $\nabla_H(2\varpi_1)$ and the other composition factors ([7, II 4.13, 4.14]), we get $\varpi_2 \in \mathcal{S}$.

As $\operatorname{res}^G_H \nabla_G(\varpi_1)$ is irreducible, we also have $\varpi_1 \in \mathcal{S}$. The rest goes as for the pair E_6, A_5A_1.

4. The pairs E_7, A_7 and E_7, D_6A_1

Say G is simply connected of type E_7 in characteristic $p > 2$ and H is the fixed point group of an involution. There are three cases, up to conjugacy. In the first case H is a Levi subgroup of a parabolic subgroup. We now consider the other two cases. One may have H of type A_7 or one may have H of type D_6A_1.

We argue as before. If H is of type D_6A_1, we show that $\varpi_1, \varpi_2, \varpi_7 \in \mathcal{S}$ by the argument with exterior tensor products used to show $\varpi_1 \in \mathcal{S}$ for the E_8, E_7A_1 pair. If H is of type A_7, we have $\varpi_1, \varpi_7 \in \mathcal{S}$ for the same reason, involving the sum formula, as the reason why $\varpi_8 \in \mathcal{S}$ for the E_8, D_8 pair. If $p = 7$, we see in the same manner that $\varpi_2 \in \mathcal{S}$. If $p \neq 7$, use $M = \nabla_G(\varpi_1) \otimes \nabla_G(\varpi_7)$ to get $\varpi_2 \in \mathcal{S}$. To get $\varpi_3 \in \mathcal{S}$ use $M = \nabla_G(\varpi_1) \wedge \nabla_G(\varpi_1)$. To get $\varpi_4 \in \mathcal{S}$ use $M = \nabla_G(\varpi_2) \wedge \nabla_G(\varpi_2)$. To get $\varpi_5 \in \mathcal{S}$ use $M = \wedge^3 \nabla_G(\varpi_7)$ if $p \neq 3$ and $M = \nabla_G(\varpi_1) \wedge \nabla_G(\varpi_2)$ otherwise. To get $\varpi_6 \in \mathcal{S}$ use $M = \nabla_G(\varpi_7) \wedge \nabla_G(\varpi_7)$.

5. The pairs F_4, B_4 and F_4, C_3A_1

Say G is of type F_4 in characteristic $p > 2$ and H is the fixed point group of an involution. There are two cases, up to conjugacy. One may have H of type B_4 or one may have H of type C_3A_1.

We argue as before. If H is of type C_3A_1, we show that $\varpi_1, \varpi_4 \in \mathcal{S}$ by the argument with exterior tensor products used to show $\varpi_1 \in \mathcal{S}$ for the E_8, E_7A_1 pair. If H is of type B_4, we have $\varpi_1, \varpi_4 \in \mathcal{S}$ for the same reason, involving the sum formula, as why $\varpi_8 \in \mathcal{S}$ for the E_8, D_8 pair. To get $\varpi_3 \in \mathcal{S}$ use $M = \nabla_G(\varpi_4) \wedge \nabla_G(\varpi_4)$. To get $\varpi_2 \in \mathcal{S}$ use $M = \nabla_G(\varpi_1) \wedge \nabla_G(\varpi_1)$.

Theorem 29. (Brundan's Conjecture [3]) *Let G be semisimple simply connected. If either*

(i) *H is the centralizer of a graph automorphism of G; or*

(ii) *H is the centralizer of an involution of G and the characteristic is at least three,*

then (G, H) is a Donkin pair.

Proof. If G is simple, we have either a Levi subgroup case, first settled in [4] (see also Remarks 9, 25), or a case treated in [3], or a case treated above, up to conjugacy.

If G is semisimple, the automorphism permutes the simple components and the problem breaks up according to the orbits in the set of simple components. Indeed suppose $G = G_1 \times G_2$ with G_i invariant under the automorphism σ. Then the fixed point group H of σ is a direct product $H_1 \times H_2$ of fixed point groups. To see if (G,H) is a Donkin pair, one needs to inspect the restriction to H of an exterior tensor product $\nabla_G(\lambda) = \nabla_{G_1}(\lambda_1) \boxtimes \nabla_{G_2}(\lambda_2)$. If both (G_i, H_i) are Donkin pairs, then one gets the desired good filtration of $\mathrm{res}^G_H(\nabla_G(\lambda))$ by combining the filtrations of the $\mathrm{res}^{G_i}_{H_i}(\nabla_{G_i}(\lambda_i))$. So apart from the case where G is simple, there is just one more interesting case. It is the case of the diagonal G inside a product $G \times \cdots \times G$. $\square$

Remark 30. To see that this theorem implies Brundan's original conjecture, one must check that there are no untreated cases in characteristic two. This follows from [1].

Remark 31. Of course we would much prefer a case-free proof, based on, say, the pairing criterion.

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