This volume is dedicated to the memory of
Professor Annamalai Ramanathan
one of the originators of Frobenius splittings

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Lectures on Frobenius splittings and B-modules

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Notes by

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Preface

These notes are based on a course given at the Tata Institute of Fundamental Research in the beginning of 1990. The aim of the course was to describe the solution by O. Mathieu of some conjectures in the representation theory of semi-simple algebraic groups. These conjectures concern the inner structure of dual Weyl modules and some of their analogues.

Recall that if G is a (connected, simply connected) semi-simple complex Lie group and B a Borel subgroup, the Borel–Weil–Bott Theorem tells that one may construct the finite dimensional irreducible G-modules in the following way. Take a line bundle $\mathcal{L}$ on the generalized flag variety G/B, such that $H^0(G/B, \mathcal{L})$ does not vanish. Then $H^0(G/B, \mathcal{L})$ is an irreducible G-module, called a dual Weyl module or an "induced module", and by varying $\mathcal{L}$ one gets all finite dimensional irreducibles.

More generally one may, after Demazure, consider the B-modules $H^0(\overline{BwB}/B, \mathcal{L})$ with $\mathcal{L}$ as above. (So one still requires that $H^0(G/B, \mathcal{L})$ does not vanish.) The "Demazure character formula" determines the character of $H^0(\overline{BwB}/B, \mathcal{L})$. It was shown by P. Polo that the B-module $H^0(\overline{BwB}/B, \mathcal{L})$ has a nice homological characterization in terms of its highest weight λ (see 3.1.10). We therefore use the notation $P(\lambda)$ for this module. The $P(\lambda)$ are generalizations of dual Weyl modules. Indeed recall that nothing is lost when restricting a rational module from G to B; inducing back up from B to G one recovers the original module (see 2.1.7).

Now the conjectures are about filtrations of the dual Weyl modules $H^0(G/B, \mathcal{L})$ or their generalizations $P(\lambda)$, for semi-simple algebraic groups in arbitrary characteristic. (Over the integers, actually.) The strongest conjecture of the series is Polo's conjecture, which says that if one twists a $P(\lambda)$ by an anti-dominant character the resulting B-module can be filtered with subsequent quotients $P(\mu_i)$. In Polo's terminology—which we will follow—the twisted module has an *excellent filtration*. (In Mathieu's terminology the twisted module is *strong*.)

This conjecture, now a theorem of Mathieu, has many nice consequences. For instance, suppose one takes a semi-simple subgroup L of G corresponding with a subset of the set of simple roots. Then if one restricts the representation $P(\lambda)$ from B to $B \cap L$, that restriction has excellent filtration. For the case of dual Weyl modules this confirms Donkin's conjecture that the restriction to L of a dual Weyl module has "good filtration", *i.e.* a filtration whose successive quotients are dual Weyl modules again. (Unlike the preceding statements, this is not interesting in the case of semi-simple Lie groups, where any finite dimensional L-module has good filtration, because of complete reducibility.)

Another consequence is a solution of the well-known problem of showing that the tensor product of two modules with good filtration has good filtration. This problem was around at least since 1977 when J.E. Humphreys was drawing attention to it. Actually Mathieu has to solve this problem first, before settling Polo's conjecture. Mathieu's proof was the first that did not need to exclude any cases. (And this was achieved by not having any case distinctions to begin with.) Later a proof has been found that uses the canonical bases of Lusztig (= crystal bases of Kashiwara).

A different type of consequence, amply demonstrated in the works of Donkin, is that many results can be carried over from characteristic 0 to characteristic p. This is because modules with excellent filtration have nice cohomological properties and thus nice base change properties. (But observe that the proofs by Mathieu start at the other end and rely very much on characteristic p methods.)

Although the subject of the course is the contribution of Mathieu, one should of course not forget the work of Wang, Donkin, Polo, ... that prepared the way. This story is not told here. To exacerbate things, but in keeping with established practice, our choice of names of mathematicians in terminology is quite arbitrary. We encourage the reader to check the references for all the things that are left out.

As is already evident from the above, we place much emphasis on B-modules (more than Mathieu did). Indeed we believe a good setting for the theory is provided by the category of B-modules, enriched with the tensor product operation and also with a *highest weight category structure* (in the sense of Cline, Parshall, Scott [2]) with the $P(\lambda)$ as "induced modules". In the lectures the highest weight category structure was simply disguised as a particular total ordering of the weights, dubbed "length–height order". (Weights are ordered by length according to a Weyl group invariant inner product, and then for fixed length by height.) Indeed no derived categories are found in the notes.

In [35] we identified another class of B-modules. The module in this class with highest weight μ we call $Q(\mu)$. It is related to the $P(\lambda)$ by the following type of duality:

$$\dim(\operatorname{Ext}^i_B(Q(\mu)^*, P(\lambda))) = \begin{cases} 1, & \text{if } i = 0 \text{ and } \lambda = -\mu; \\ 0, & \text{otherwise.} \end{cases}$$

The interaction between the $P(\lambda)$ and the $Q(\nu)$ has much relevance for the filtration conjectures.

Mathieu's proof of these conjectures involves an innovative way to exploit Frobenius splittings on Bott-Samelson-Demazure-Hansen resolutions of Schubert varieties and some of their generalizations. It was interesting to be lecturing about Frobenius splittings at TIFR, with the originators of that theory in the audience.

Warning. When we speak of highest weight, we are using the ordering in which the roots of B are positive. This is opposite to the choice in much of the recent literature, but we hope the reader agrees that in our situation—where the main concern is modules $P(\lambda)$ with one-dimensional socles generated by a highest weight vector of weight λ—it would be silly to reverse the ordering.

The lectures given in Bombay have served as a starting point for the present notes, but it was not a straightforward job to convert the oral story into something organized and intelligible. I am very grateful to S. P. Inamdar who wrote the main body of the notes. He smoothed out many rough spots and mercifully removed some of my less fortunate variations.

Finally, it is a pleasure to thank colleagues and staff at TIFR for providing such a friendly environment for us visitors.

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Chapter 1

Preliminaries

This chapter should be taken as a guideline of what notation and terminology is used later on during the course rather than giving a complete treatment of the structure theory of reductive groups. An excellent reference for a detailed discussion of the contents of the first section is the book [Humphreys: Linear Algebraic Groups]. Indeed, most of the material is taken from it.

1.1 Reductive Algebraic Groups

Let k be an algebraically closed field. Let G be a variety over k with the structure of a group on its set of points. We call G an *algebraic group* if the maps $\mu : G \times G \to G$, where $\mu(x, y) = xy$, and $\tau : G \to G$, where $\tau(x) = x^{-1}$, are algebraic morphisms.

By a morphism of groups we mean an algebraic group homomorphism between the two varieties. A morphism from an algebraic group G to $\mathrm{GL}(n, k)$ is called a (rational) representation of G of dimension n with underlying vector space k^n.

The *additive group* $\mathbb{G}_a$ is the affine line $\mathbb{A}^1$ with the group law $\mu(x, y) = x + y$. The *multiplicative group* $\mathbb{G}_m$ is the open affine subset $k^* \subset \mathbb{A}^1$ with group law $\mu(x, y) = xy$. The set $\mathrm{GL}(n, k)$ of $n \times n$ invertible matrices with entries in k is a group under matrix multiplication called the *general linear group*.

A closed subgroup of an algebraic group is an algebraic group. Thus the *special linear group* $\mathrm{SL}(n, k)$ of all the matrices of determinant 1 in $\mathrm{GL}(n, k)$

and the subgroup $\mathrm{D}(n,k)$ of all diagonal matrices are algebraic groups. An algebraic group is called a torus if it is isomorphic to $\mathrm{D}(n,k)$ for some n.

Let G be an algebraic group, X a variety. We say that G *acts* (rationally) on X if we are given a morphism $\varphi : G\times X \to X$ such that for $x_i \in G,\ y \in X$ we have $\varphi(x_1, \varphi(x_2, y)) = \varphi(x_1x_2, y)$ and $\varphi(e, y) = y$. One usually writes $g \cdot v$ or gv for $\varphi(g, v)$.

Let $\varphi : G \to \mathrm{GL}(n,k)$ be a (rational) representation of an algebraic group G. Then G acts on the affine n-space $\mathbb{A}^n$ via this representation, *i.e.* $x \cdot v = \varphi(x)(v)$, and thus on a n-dimensional vector space V over k. In this case we call V a (rational) G-module. More generally, if G acts linearly on a k-vector space V, then V is called a (rational) G-module if it is the union of finite dimensional subspaces on which G acts rationally.

A *character* of an algebraic group G is a morphism of algebraic groups $\chi : G \to \mathbb{G}_m$. We denote the group of characters of G by $X(G)$.

Let H be a diagonal subgroup (or a subgroup of $\mathrm{GL}(n,k)$ which is diagonalisable). Let V be an H-module. Then V decomposes as direct sum of subspaces V_α, where α runs over the the character group $X(H)$ of H and

$$V_\alpha = \{\, v \in V \mid x \cdot v = \alpha(x)v \,\}.$$

Those α for which V_α is non-zero are called the *weights* of V and $v \in V_\alpha$ is called a weight vector of weight α.

Every algebraic group contains a unique largest connected normal solvable group. We call this subgroup of G the *radical* of G. It is denoted by $R(G)$. A group G is called semi-simple if $R(G)$ is trivial. The subgroup of $R(G)$ consisting of all unipotent elements is normal in G; we call it the *unipotent radical* of G. We denote it by $R_u(G)$. We call G reductive if $R_u(G)$ is trivial.

The group $\mathrm{SL}(n,k)$ is semi-simple and $\mathrm{GL}(n,k)$ is reductive. Note that any semi-simple group is automatically reductive.

From now on we will assume that our group G is connected reductive.

A *Borel subgroup* of G is a maximal closed connected solvable subgroup of G. A connected solvable subgroup of largest possible dimension in G is of course a Borel subgroup and it is also true that every Borel subgroup of G has the same dimension. In fact we have the following stronger theorem:

Theorem 1.1.1 *Let B be any Borel subgroup of G. Then G/B is a projective variety, and all other Borel subgroups are conjugate to B.*

We call a closed subgroup of G *parabolic* if it contains a Borel subgroup. The centralizer C of a maximal torus T of G is called a *Cartan subgroup* of G. Note that we did not put the condition of it being a connected subgroup of G as it can be shown that any Cartan subgroup of a connected algebraic group is connected. For reductive groups, the Cartan subgroup $C_G(T)$ equals T.

We now state the Borel Fixed Point Theorem and some of its consequences.

Theorem 1.1.2 (Borel Fixed Point Theorem) *Let B be a connected solvable algebraic group, and X be a complete variety on which B acts. Then B has a fixed point in X.*

From this theorem one can deduce Theorem 1.1.1 and also:

(i) All maximal tori, and all Borel subgroups are conjugate.

(ii) P is parabolic subgroup of G if and only if G/P is a complete variety.

If S is any torus in G, we call $N_G(S)/C_G(S)$ *Weyl group of G relative to S*, where $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer in G of a subgroup H of G. Since all maximal tori are conjugate, all their Weyl groups are isomorphic. We call this group the *Weyl group of G*. We denote it by W. We state here some of the important properties of this group W. Recall that G is a connected reductive algebraic group.

(i) W is a finite group.

(ii) W is generated by elements s_i $(1 \leq i \leq l)$, for some l, with the following defining relations between them:
$(s_i s_j)^{m(i,j)} = e$, with $m(i,i) = 1$ and $2 \leq m(i,j) < \infty$ for $i \neq j$. A group generated by elements having such defining relations is called a *Coxeter group.*

(iii) If $\chi \in X(T)$ and $t \in T$ the formula

$$(w\chi)(t) = \chi(n^{-1}tn)$$

gives us an action of an element $w \in W$ on $X(T)$; here n denotes a coset representative of w in $N_G(T)$.

(iv) Since the real vector space $X(T) \otimes \mathbb{R}$ is a W-module, we can put a metric on it which is invariant under the action of the finite group W, *i.e.* there is an inner product (,) such that $(w\chi, w\mu) = (\chi, \mu)$ for every $\chi, \mu \in X(T)$. Put $l(\mu) = (\mu, \mu)^{1/2}$.

(v) If we fix a Borel subgroup B and a maximal torus $T \subset B$, we get a preferred set of generators of W. We call them simple reflections. If they are indexed by a (finite) set I (e.g. the nodes of the Dynkin diagram), then for each $i \in I$, we also have a simple root α and we may choose a homomorphism $\mathrm{SL}(2, k) \to G$, mapping

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(t), \qquad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-\alpha}(t).$$

Here if β is a root, $x_\beta : \mathbb{G}_a \to B$ denotes a conveniently normalized injective homomorphism satisfying $hx_\beta(t)h^{-1} = x_\beta(\beta(h)t)$ for $t \in k$, $h \in T$. (Cf. [34; Chapters 9, 10].) Our homomorphism $\mathrm{SL}(2, k) \to G$ has the property that it has at most $\{1, -1\}$ as kernel and hence the image is isomorphic to either $\mathrm{SL}(2, k)$ or to the quotient $\mathrm{PSL}(2, k)$ of $\mathrm{SL}(2, k)$ by this subgroup of order 2. We note that in characteristic 2, the above group $\{1, -1\}$ does not differ from $\{1\}$ and one must replace it by a “group scheme” of order 2.

If $\varphi : G \to \mathrm{GL}(V)$ is a representation, the *weights* of V are the images in $X(T)$ of the weights of $\varphi(T)$ in V via the canonical homomorphism $X(\varphi(T)) \to X(T)$. We make W act on weights of V via this canonical homomorphism.

Let us fix a Borel subgroup B and a maximal torus T of B. Let W denote the Weyl group of G relative to T. As we have just pointed out this choice of B and T gives us a preferred set of generators of W and for each simple reflection we either have a copy of $\mathrm{SL}(2, k)$ or $\mathrm{PSL}(2, k)$ embedded in G. Any such subgroup together with B generates a parabolic subgroup of G. We call these subgroups *minimal parabolic* subgroups of G. If s_i is a simple reflection in W and P_i denotes the associated minimal parabolic subgroup then P_i contains a representative of s_i in G. Note that since T lies in B, the double coset BnB is independent of the choice of n representing a given $w \in W$. We thus write BwB for this double coset. Its image in G/B is called a *Bruhat cell* and the closure of a Bruhat cell is called a *Schubert variety*.

It is a union of Bruhat cells. Any element $w \in W$ can be expressed as the product $s_1 \cdots s_r$ for some sequence $\{s_1, \ldots, s_r\}$ of simple reflections in W. If this expression is reduced and P_i is the minimal parabolic corresponding with s_i, then BwB has as its closure the set $P_1 \cdots P_r$, *i.e.* the image of $P_1 \times \cdots \times P_r$ under the multiplication map $G \times \cdots \times G \to G$.

Theorem 1.1.3 (Bruhat decomposition) *For any reductive group G, we have $G = \bigcup_{w \in W} BwB$, with $Bw_1B = Bw_2B$ if and only if $w_1 = w_2$ in W.*

Corollary 1.1.4 *Let G be a reductive group and B be a Borel subgroup of G. We have $G/B = \bigcup_{w \in W} BwB/B$ with $Bw_1B/B = Bw_2B/B$ if and only if $w_1 = w_2$.*

This decomposition gives a stratification of the smooth projective variety G/B by the Bruhat cells, the i^{th} stratum being the union of all Bruhat cells of dimension i. A codimension one Schubert variety of G/B is called a Schubert divisor of G/B.

1.2 Demazure Desingularisation of G/B

The projective variety G/B being homogeneous it is smooth. However, the Schubert varieties are not all smooth subvarieties of G/B. Further, two Schubert divisors need not intersect transversally with each other. Demazure constructed a "desingularisation" of G/B to overcome this problem. In this section we first discuss Kempf's approach via the standard modifications. Next we reformulate the resolution in terms of fibre bundles. It is the latter description which will be used later.

Recall G is a connected semi-simple or reductive algebraic group over an algebraically closed field of arbitrary characteristic. We fixed a maximal torus T and a Borel subgroup B containing T. The unipotent radical of B will be denoted U. If W is the Weyl group of G, then we have a preferred set of generators of W, called simple reflections. We typically denote them by s or s_i. Then P_s or P_i denotes the associated (minimal) parabolic subgroup of G. For any parabolic subgroup $Q \supseteq B$ of G, by a Schubert variety in G/Q we mean the closure of a B-orbit in G/Q. We will be dealing mostly with

Schubert varieties in G/B. The properties for Schubert varieties in G/Q can be deduced from those of in G/B by studying the fibration $G/B \to G/Q$.

We have the Bruhat decomposition $G/B = \bigcup_{w \in W} BwB/B$ of G/B into B-orbits. Note that as this is a finite union, any B-invariant irreducible closed subvariety of G/B is a Schubert variety.

Let $X_w = \overline{BwB}/B$ be a Schubert variety of dimension r. Let $w = s_1 \cdots s_r$ be a reduced expression for w. We also complete it into a reduced expression for the element w_N of maximal length: $w_N = s_1 \cdots s_r \cdots s_N$. Let $w_j = s_1 \cdots s_j$ and $X_j = X_{w_j}$ be the corresponding Schubert variety of dimension j. Note that $X_r = X_w$. It is known (refer to Kempf [13]) that the varieties X_j are saturated for the morphism $\pi_j : G/B \to G/P_j$ and that X_{j-1} maps birationally onto its image $\pi_j(X_{j-1}) = \pi_j(X_j)$.

The *standard modification* $\varphi_j : M_j \to X_j$ is defined by the Cartesian square:

$$\begin{array}{ccc} M_j & \xrightarrow{\varphi} & X_j \\ \downarrow & & \downarrow \pi_j \\ X_{j-1} & \xrightarrow{\pi_j} & \pi_j(X_{j-1}) \end{array}$$

Thus M_j is a $\mathbb{P}^1$-bundle over the divisor X_{j-1} in X_j.

The *Demazure resolutions* (or desingularisations) $\psi_j : Z_j \to X_j$ are defined inductively. We start by taking $Z_0 = X_0$, a point, and $\psi_0 : Z_0 \to X_0$ the identity morphism. Then $\psi_j : Z_j \to X_j$ is defined by the diagram with Cartesian squares:

$$\begin{array}{ccccc} Z_j & \longrightarrow & M_j & \xrightarrow{\varphi} & X_j \\ \downarrow f_j & & \downarrow & & \downarrow \pi_j \\ Z_{j-1} & \xrightarrow{\psi_{j-1}} & X_{j-1} & \xrightarrow{\pi_j} & \pi_j(X_{j-1}) \end{array}$$

Note that $f_j : Z_j \to Z_{j-1}$ is a $\mathbb{P}^1$-bundle being a pullback of the $\mathbb{P}^1$-bundle $X_j \to \pi_j(X_{j-1})$. This implies that the Z_j are nonsingular by induction. We also have a section $\sigma_j : Z_{j-1} \to Z_j$ given by the inclusion $X_{j-1} \subset X_j$. Further ψ_j is birational since π_j is birational on X_{j-1} and by inductive hypothesis we can assume ψ_{j-1} is birational.

Since $X_r = X_w$ we get by this process a standard modification and Demazure resolution of X_w. Note that this resolution depends on the reduced expression chosen for w. We also get a Demazure resolution for G/B by this process as $X_{w_N} = G/B$.

We now prepare to give another description for the varieties obtained by the desingularisation process. Recall that since $G \to G/B$ is a principal B-fibration, given any B-space X (*i.e.* any variety X such that B acts on it on the left) we can associate a fibre bundle over G/B with fibre being isomorphic to X. We denote such associated fibre bundle by $G \times^B X$. It is defined as the quotient of $G \times X$ given by the equivalence relation $(g, x) \sim (gb, b^{-1}x)$. Note that the natural left multiplication action of G on $G \times X$ descends to a left action on the associated fibre bundle. This action commutes with the projection morphism and thus the associated fibre bundle is a G fibre bundle on G/B.

Exercise 1.2.1 (i) This fibre bundle is locally trivial in the Zariski topology. (Check that for any $g \in G$ it is trivial over gBw_0B/B, where w_0 denotes the longest element of the Weyl group.)

(ii) Prove similar statements for $P \times^B X$ and for $G \times^P Y$ where P is a parabolic—always containing B—and Y is a P-space. Here it may help to be familiar with standard coordinates in Bruhat cells, as explained for instance in [34; Chapter 10]. Observe that the fibration $G/B \to G/P$ is an example of an associated fibre bundle with $X = P/B$.

Remark 1.2.2 If X is in fact a G-space, then the fibre bundle $G \times^B X$ is globally trivial by means of the map $G \times^B X \to G/B \times X$ which sends the class of (g, x) to (gB, gx).

Now each parabolic P_i contains B, and hence they are B-invariant under the left translation action of B on G. The Demazure desingularisation of X_w is the associated fibre bundle $Z_r = P_1 \times^B (P_2 \times^B \cdots \times^B P_r/B)$ and the map $\psi_r : Z_r \to X_w$ is the multiplication map defined on the product $P_1 \times \cdots \times P_r$ which actually descends to the associated fibre bundle. This description will be very useful for us later on. It now enables us to say that in the Bruhat decomposition of G/B, the varieties $\overline{BwB}/B$ are birational to the image of $P_1 \times^B (P_2 \times^B \cdots \times^B P_r/B)$ under the multiplication map. Thus the dimension of X_w is just the length of the reduced expression of w. More specifically, the subset $Bs_1B \times^B (Bs_2B \times^B \cdots \times^B Bs_rB/B)$ of Z_r maps isomorphically to the Bruhat cell BwB/B. (Compare [34; Chapter 10].) It will also be useful to consider the analogue of Z_r for words $s_1 \cdots s_r$ that are not reduced. Then of course one will not get a birational map.

To any B-module M, we can associate a fibre bundle on G/B with the fibre being isomorphic with M as before. We denote this bundle on G/B by $\mathcal{L}(M)$. This G fibre bundle is called the associated vector bundle for the given representation. The reader will see during the course of lectures that this construction will enable us to use "geometric" results to study the representations of G and B.

Chapter 2

B-Module Theory

Let k be an algebraically closed field. Let H be an algebraic group over k. Let V be vector space over k. A group morphism $H \to GL(V)$ is called a (rational) representation of H. When G is reductive and connected we get a good hold on the representation theory of G by looking at the representations of its Borel subgroup B. For example, Weyl's highest weight theory in characteristic zero gives a description of irreducible representations of G in terms of dominant characters of B. In this chapter we introduce the dual Joseph modules and relative Schubert modules. These two classes of B-modules are analogues of irreducible G-modules in characteristic zero.

In the first section we prove the Frobenius reciprocity for our connected reductive group G and its Borel subgroup B. Let $\mathcal{C}_G$ denote the category of G-modules. The reciprocity implies that $\mathcal{C}_G$ is a full subcategory of $\mathcal{C}_B$.

In the second section, we introduce the Joseph functor H_w on the category of B-modules associated to a Schubert variety $X_w \subset G/B$.

In the third section we introduce the dual Joseph modules. For a character λ, let $w = w_\lambda$ denote the minimal element of the Weyl group W such that $w^{-1}\lambda \in X(T)^- = \{\, \mu \in X(T) \mid (\mu, \alpha) \leq 0 \text{ for all roots } \alpha \text{ of } B \,\}$. The dual Joseph module $P(\lambda)$ is then defined as $H_w(w^{-1}\lambda)$. The relative Schubert module $Q(\lambda)$ is defined as the kernel of the restriction map from $H_w(w^{-1}\lambda)$ to the sections over the boundary ∂X_w of X_w.

In positive characteristic we do not have complete reducibility. In order to "understand" the indecomposable B-modules we introduce the concepts of excellent filtrations and relative Schubert filtrations. Indeed we will be studying the excellent filtrations extensively throughout these notes.

We finish this chapter by giving examples of modules with relative Schubert filtration.

2.1 Frobenius Reciprocity

Let H be an algebraic group. We call an H-module M *simple* (and the corresponding representation *irreducible*) if $M \neq 0$ and if M has no H-submodules other than 0 and M. It is called *indecomposable* if it cannot be decomposed into a direct sum of two proper H-submodules and it is *semi-simple* if it is a direct sum of simple H-submodules. For any M the sum of all its simple submodules is called the *socle* of M and denoted by $\operatorname{soc}_H M$ or simply $\operatorname{soc} M$ if it is clear which H is considered. It is the largest semi-simple H-submodule of M. Each one-dimensional H-module is simple. Let $\mathcal{C}_B$ and $\mathcal{C}_G$ denote the categories of B-modules and G-modules respectively.

For a subgroup H of G and a G-module M we can restrict the action of G to H. This functor from $\mathcal{C}_G$ to $\mathcal{C}_H$ is called the *restriction functor* and denoted by $\operatorname{res}^G_H(?)$. It takes an exact sequence of G-modules to an exact sequence of H-modules and thus it is an exact functor.

Let G be our reductive connected algebraic group. We fix once and for all a maximal torus T and a Borel subgroup B of G containing T. Let W be the Weyl group of G. Recall that our choice of B gives us a set of preferred generators $S = \{s_1, \dots, s_l\}$ of W, called simple reflections. Let $X(T)$ denote the set of characters of T. Recall that the Weyl group W acts naturally on characters of T and fix a W-invariant inner product on $X(T) \otimes \mathbb{R}$.

Since $T \subset B$, $\operatorname{res}^B_T(M)$ is a T-module for any B-module M. As T is diagonalisable, M then decomposes as a direct sum of one-dimensional submodules. The character with which T acts on a one-dimensional submodule is called the *weight* of that submodule. The direct sum of the one-dimensional submodules of M having the same weight λ is called the weight space M_λ of M. Let $\mathcal{C}_{\leq R}$ denote the category of B-modules all of whose weights are of length not more than R with respect to the chosen W-invariant inner product on $X(T) \otimes \mathbb{R}$. For a B-module M, we denote by $M_{\leq R}$ the largest B-submodule of M which is in $\mathcal{C}_{\leq R}$. This defines a left exact functor from $\mathcal{C}_B$ to $\mathcal{C}_{\leq R}$. For example, if $R = 0$, then $M_{\leq R}$ is nothing else than $H^0(B, M)$, the subspace of B-fixed vectors in M.

Exercise 2.1.1 Give an example to show that the functor $M \mapsto M_{\leq R}$ is not right exact.

For $M \in \mathcal{C}_B$, let $\mathcal{L}(M)$ denote the associated G-vector bundle, (possibly infinite dimensional), on G/B, as introduced before. The group G acts on $\mathcal{L}(M)$ and therefore we have a natural G action on $H^0(G/B, \mathcal{L}(M))$, cf. Jantzen [11; I 5.11 Remark]. We call this G-module $\mathrm{ind}_B^G(M)$. Thus we have a functor $\mathcal{C}_B \to \mathcal{C}_G$ given by $M \mapsto \mathrm{ind}_B^G(M)$. This functor is called the *induction functor*. The reader should note that in Jantzen's book the induction functor is defined more algebraically but for us this equivalent definition will prove more useful.

If M were a G-module then the associated vector bundle $\mathcal{L}(M)$ is isomorphic with the trivial bundle $G/B \times M$. Further, as G/B is a complete variety we have $H^0(G/B, \mathcal{L}(M)) = M$. Therefore if $M \in \mathcal{C}_G$, then $\mathrm{ind}_B^G(M) = M$.

Remark 2.1.2 If P is a parabolic subgroup of G then we define in a similar way the induction functor $\mathrm{ind}_B^P(?)$ by assigning the P-module $H^0(P/B, \mathcal{L}(M))$ to a B-module M. As before, if M were a P-module, we get $\mathrm{ind}_B^P(M) = M$.

Remark 2.1.3 The fibre over the B-fixed point B/B of the vector bundle $\mathcal{L}(M)$ is canonically isomorphic with M. Therefore the restriction map $H^0(G/B, \mathcal{L}(M)) \to \mathcal{L}(M)|_{B/B}$ gives a natural B-equivariant morphism $\mathrm{ind}_B^G(M) \to M$. This map is called the *evaluation map*.

Exercise 2.1.4 (i) Prove that the evaluation map $\mathrm{ind}_B^G(M) \to M$ is an isomorphism if M is a G-module.

(ii) Give examples to prove that this map need not be injective and need not be surjective.

Remark 2.1.5 The functor $\mathrm{ind}_B^G(?)$ is left exact and commutes with forming direct sums, intersections of submodules, and direct limits over directed systems. (The latter property helps to understand the meaning of $\mathrm{ind}_B^G(M)$ for an infinite dimensional module M, as M is a union of its finite dimensional submodules.) There is a transitivity of induction, that is, if $B \subseteq P$, then $\mathrm{ind}_B^G = \mathrm{ind}_P^G \circ \mathrm{ind}_B^P$. We also have the following tensor identity:

$$\mathrm{ind}_B^G(M \otimes \mathrm{res}_B^G(N)) = (\mathrm{ind}_B^G(M)) \otimes N$$

for any G-module N and B-module M.

The Frobenius reciprocity says that the induction functor is right adjoint of the restriction functor.

Proposition 2.1.6 (Frobenius reciprocity) *For any G-module N and B-module M we have* $\mathrm{Hom}_G(N, \mathrm{ind}_B^G(M)) = \mathrm{Hom}_B(\mathrm{res}_B^G(N), M)$.

Proof: Composing $N \to \mathrm{ind}_B^G(M)$ with the evaluation map $\mathrm{ind}_B^G(M) \to M$ gives us a natural map $\mathrm{Hom}_G(N, \mathrm{ind}_B^G(M)) \to \mathrm{Hom}_B(\mathrm{res}_B^G(N), M)$. Conversely given a B-equivariant map $f : N \to M$ we associate to it a G-equivariant map $\widetilde{f} : N \to \mathrm{ind}_B^G(M)$ by the formula $\widetilde{f}(n) = \left(\overline{g} \mapsto \overline{(g, f(g^{-1}n))}\right)$. □

Corollary 2.1.7 *One may view $\mathcal{C}_G$ as a full subcategory of $\mathcal{C}_B$.*

Proof: If $M, N \in \mathcal{C}_G$ then $\mathrm{Hom}_G(N, M)) = \mathrm{Hom}_G(N, \mathrm{ind}_B^G \mathrm{res}_B^G M)) = \mathrm{Hom}_B(\mathrm{res}_B^G(N), \mathrm{res}_B^G M)$. □

Remark 2.1.8 As we will see, many questions about G-modules are special cases of questions about B-modules through this identification of $\mathcal{C}_G$ with a subcategory of $\mathcal{C}_B$.

Remark 2.1.9 Here G needed not be reductive, of course, and we will not hesitate to use the result more generally. We will often discuss only G and/or B, leaving it to the reader to find the scope of the arguments. When in doubt, consult [11].

In fact the identification of $\mathcal{C}_G$ with a full subcategory of $\mathcal{C}_B$ even works on the level of Ext groups. This is derived from the corollary using Kempf's Vanishing Theorem A.2.7. Indeed we have

Lemma 2.1.10 *Let P be a parabolic subgroup containing B and let M, N be P-modules. Then* $\mathrm{Ext}_P^i(M, N) = \mathrm{Ext}_B^i(M, N)$ *for all i.*

Proof: In [11; II Corollary 4.7] this is stated for G and P instead of P and B, but the argument is the same. □

2.2 Joseph's Functors

In characteristic zero, a rational representation of G is completely reducible. Further, the irreducible G-modules are induced up from irreducible B-modules. We do not have such a nice result for representations of G in characteristic $p > 0$. In this section we define Joseph's functors. These functors will then lead us to study dual Joseph modules and relative Schubert modules which form some kind of building blocks for a class of representations of B or G, sharing good properties with the G-modules of characteristic 0.

For a Schubert variety X_w of G/B, we get a natural B action on $H^0(X_w, \mathcal{L}(M))$, the sections of the vector bundle $\mathcal{L}(M)|_{X_w}$ over X_w.

Definition 2.2.1 The functors $H_w : \mathcal{C}_B \to \mathcal{C}_B$ given by the rule $M \mapsto H^0(X_w, \mathcal{L}(M))$ are called *Joseph's functors.*

Remark 2.2.2 The Joseph functors are also defined for Kac-Moody groups using cohomological algebra. (See [18].) They are actually dual to the functors originally studied by Joseph in [12], also with cohomological algebra. The above definition gives a kind of "representability" of the Joseph Functors.

Remark 2.2.3 It should be noted that for the element of largest length w_0 of W, the two functors H_{w_0} and ind_B^G are the same. (Up to res_B^G, which may safely be ignored from now on, because of Corollary 2.1.7.)

Remark 2.2.4 We denote by P_s the minimal parabolic subgroup associated to a simple reflection $s \in S$. The Schubert variety $X_s \subset G/B$ is the image of P_s under the projection map. It is thus isomorphic with the complete variety P_s/B. Also, for any B-module M the vector bundle $\mathcal{L}(M)$ on P_s/B is isomorphic with the restriction of the vector bundle $\mathcal{L}(M)$ to $X_s \subset G/B$. We thus get that the functor $H_s : \mathcal{C}_B \to \mathcal{C}_B$ is the composition of two functors $\mathrm{res}_B^{P_s} \circ \mathrm{ind}_B^{P_s}$. That is, in this particular case, the module $H_s(M)$ is a P_s-module viewed as a B-module.

Recall that for an element $w \in W$, the length $l(w)$ of w is the length of any of its reduced expressions in the chosen generators. It is independent of which reduced expression one is using and thus defines a integer valued function on W. For any $w \in W$ and $s \in S$, the preferred set of generators, we have: $l(sw) \neq l(w)$ and in fact $l(sw)$ is either $l(w)+1$ or $l(w)-1$.

Proposition 2.2.5 *For $s \in S$, $w \in W$ and $M \in \mathcal{C}_B$, we have:*

(i) $H_s H_w(M) = H_w(M)$ *if* $l(s.w) = l(w) - 1$.

(ii) $H_s H_w(M) = H_{sw}(M)$ *if* $l(s.w) = l(w) + 1$.

Proof: Let P_s denote the parabolic subgroup associated to the simple reflection $s \in W$. Consider the following diagram

$$\begin{array}{ccc} P_s \times^B X_w & \stackrel{m}{\to} & P_s X_w \subset G/B \\ \downarrow \pi & & \\ P_s/B & & \end{array}$$

where the morphism m is the multiplication map which descends to the fibre bundle.

(i): $l(s.w) = l(w) - 1$.

In this case the image of the multiplication map m is $BsBX_w \cup BX_w = X_{s.w} \cup X_w = X_w$ by [9; 28.3]. Therefore the natural left action of P_s on G/B leaves X_w invariant. The vector bundle $\mathcal{L}(M)$ on G/B has a natural G (and hence P_s) action. This gives a natural P_s action on $H^0(X_w, \mathcal{L}(M))$. When restricted to B, this action gives the B-module action on $H_w(M)$. Therefore we have $H_w(M) \in \mathcal{C}_{P_s} \subset \mathcal{C}_B$. Hence $\mathrm{ind}_B^{P_s}(H_w(M)) = H_w(M)$. Also we have $H_s(M) = \mathrm{res}_B^{P_s} \circ \mathrm{ind}_B^{P_s}(M)$. Thus by Remark 2.1.2 we get that $H_s H_w(M) = H_w(M)$.

(ii): $l(s.w) = l(w) + 1$.

Now the associated fibre bundle over P_s/B in the above diagram is such that the multiplication morphism m is birational and proper with $P_s X_w = X_{sw}$. As X_{sw} is normal (cf. [25]), this implies $m_* \mathcal{O}_{P_s \times^B X_w} = \mathcal{O}_{X_{sw}}$ ([11; II Lemma 14.5]). For a B-module M, this gives

$$\begin{aligned} H_{sw}(M) &= H^0(X_{sw}, \mathcal{L}(M)) \\ &= H^0(X_{sw}, m_* \mathcal{O}_{P_s \times^B X_w} \otimes \mathcal{L}(M)) \\ &= H^0(P_s \times^B X_w, m^* \mathcal{L}(M)) \\ &= H^0(P_s/B, \pi_* m^* \mathcal{L}(M)). \end{aligned}$$

But we have: $\pi_* m^*\mathcal{L}(M) = \mathcal{L}(H^0(X_w, \mathcal{L}(M)))$. Therefore we get that:

$$\begin{aligned} H_{sw}(M) &= H^0(P_s \times^B X_w, m^*\mathcal{L}(M)) \\ &= H^0(P_s/B, \mathcal{L}(H^0(X_w, \mathcal{L}(M)))) \\ &= H_s(H_w(M)). \end{aligned}$$

This proves the proposition. □

Exercise 2.2.6 Prove that $\pi_* m^*\mathcal{L}(M) = \mathcal{L}(H^0(X_w, \mathcal{L}(M)))$.

Corollary 2.2.7 *Let $w \in W$ and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression. Then $H_w = H_{s_{i_1}} \circ \cdots \circ H_{s_{i_r}}$.*

Let k_λ denote the one-dimensional B-module on which B acts via a character λ. We denote by $\mathcal{L}(\lambda)$ its associated line bundle and by $H_w(\lambda)$ its image under the Joseph functor $H_w(?)$.

Extra hypothesis 2.2.8 For all the questions we are interested in, one may easily reduce to the case that the commutator subgroup of our connected reductive algebraic group G is simply connected. This implies that for each simple root, the corresponding homomorphism from SL_2 into G is a closed embedding. (Recall that the other possibility would be that the image of this homomorphism is isomorphic to PSL_2.) Let us assume simply connectedness from now on. Then any line bundle on G/B is associated to a one-dimensional representation of B (cf. Corollary A.4.3) and if the associated character λ is anti-dominant *i.e.* $\lambda \in X(T)^-$, then $\mathcal{L}(\lambda)$ is base point free, *i.e.* given any point $x \in G/B$ there exists a global section $s \in H^0(G/B, \mathcal{L}(\lambda))$ with $s(x) \neq 0$. Conversely, if $H^0(G/B, \mathcal{L}(\lambda)) \neq 0$ then $\mathcal{L}(\lambda)$ is base point free (because of equivariance) and λ is anti-dominant. (See [11; II 2.6], keeping in mind that his dominant weights are our anti-dominant ones.)

Lemma 2.2.9 *For any $\lambda \in X(T)^-$, the socle of $H_w(\lambda)$ is one-dimensional and its character is $w\lambda$.*

Proof: The Bruhat decomposition of G/B says that the B-orbit of w in X_w is open (and thus dense) in X_w. Therefore for a B-module M a section of $H_w(M)$ on which B acts by a character is determined uniquely by its image under the restriction map $H_w(M) \to \mathcal{L}(M)|_w$. Therefore, since the fibre of $\mathcal{L}(\lambda)$ is of dimension one, we can have only one B-invariant (up to scalar multiplication) section of $H_w(\lambda)$. Further, as the restriction is T-equivariant, B acts by the character $w\lambda$ on such a section. On the other hand $H_w(\lambda) \neq 0$ because the line bundle is base point free. By the Borel Fixed Point Theorem, (Theorem 1.1.2) there exists a fixed point for the B action on the projective space $\mathbb{P}(H_w(\lambda))$. This proves the existence (cf. [11; II 2.1]) of a B-invariant one-dimensional subspace of $H_w(\lambda)$. Thus the result. □

Corollary 2.2.10 *Let $\lambda \in X(T)^-$. Then, $w'\lambda$ occurs as a weight in $H_w(\lambda)$ for every $w' \leq w$ in the Bruhat order.*

Proof: Since the line bundle $\mathcal{L}(\lambda) = \mathcal{L}(k_\lambda)$ is base point free, the natural restriction map from $H_w(\lambda)$ to $H_{w'}(\lambda)$ is not the zero map. The socle of the image is thus a non-zero subspace of the socle of $H_{w'}(\lambda)$. The socle of $H_{w'}(\lambda)$ is of dimension one and has weight $w'\lambda$. Therefore as the restriction map is B-equivariant, $w'\lambda$ occurs as a weight in $H_w(\lambda)$. □

Lemma 2.2.11 *For any two B-invariant closed subsets S, S' of G/B and any line bundle without base points $\mathcal{L}$ on G/B, we have an exact Mayer–Vietoris sequence*

$$0 \to H^0(S \cup S', \mathcal{L}) \longrightarrow H^0(S, \mathcal{L}) \oplus H^0(S', \mathcal{L}) \longrightarrow H^0(S \cap S', \mathcal{L}) \to 0$$

Moreover the map $H^0(G/B, \mathcal{L}) \to H^0(S, \mathcal{L})$ is surjective.

Proof: This Mayer–Vietoris Lemma uses Ramanathan [31] for the surjectivity statements (cf. Proposition A.2.6), and it uses Ramanathan once more for knowing that $S \cap S'$ is also the scheme theoretic intersection, *i.e.* that its ideal sheaf in G/B is the sum of the ideal sheafs of S and S'. This then gives an exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{S \cup S'} \longrightarrow \mathcal{I}_S \oplus \mathcal{I}_{S'} \longrightarrow \mathcal{I}_{S \cap S'} \longrightarrow 0$$

from which the result follows easily. (The “unattentive” reader is alerted here that one should be worrying that the scheme theoretic intersection might not be reduced. See the exercise below.) □

Remark 2.2.12 The similar Mayer–Vietoris exact sequence is valid on G/P for any parabolic P, cf. Exercise A.2.9. The passage from G/B to G/P is easy as the projection $G/B \to G/P$ is a P/B fibration.

Exercise 2.2.13 Find an example of an affine variety X and two closed subvarieties S, S' so that $H^0(S \cup S', \mathcal{O}_X)$ is not the kernel of $H^0(S, \mathcal{O}_X) \oplus H^0(S', \mathcal{O}_X) \to H^0(S \cap S', \mathcal{O}_X)$. Here unions and intersections are simply taken set theoretically.

Definition 2.2.14 We say a weight occurring in an indecomposable B-module is *extremal* if it has the largest length.

The modules $H_w(\lambda)$ are indecomposable as they have one-dimensional socle. The following proposition gives a nice description of the extremal weights of $H_w(\lambda)$.

Proposition 2.2.15 *Let $\lambda \in X(T)^-$. The extremal weights of $H_w(\lambda)$ are $w'\lambda$ for $w' \leq w$. Further, the weight spaces corresponding to the extremal weights are one-dimensional.*

Proof: The Corollary 2.2.10 says that these $w'\lambda$ occur as a weight in $H_w(\lambda)$.

For $\lambda \in X(T)^-$ the global sections module $H^0(G/B, \mathcal{L}(\lambda))$ is $\neq 0$. We start with showing that the extremal weights of $H_{w_0}(\lambda) := H^0(G/B, \mathcal{L}(\lambda))$ are $w\lambda$ for $w \in W$.

The module $H^0(G/B, \mathcal{L}(\lambda))$ is a G-module. Therefore for every $w \in W$ and every extremal weight ν, the character $w\nu$ occurs as a weight of $H^0(G/B, \mathcal{L}(\lambda))$. Further $w\nu$ is also extremal as the inner product on the vector space $X(T) \otimes \mathbb{R}$ is W-invariant. Let $w_\nu \in W$ be such that the character $\nu_0 = w_\nu \nu$ is a dominant character, *i.e.* such that $\nu_0 \in X(T)^+ = \{ \mu \in X(T) \mid (\mu, \alpha) \geq 0$ for all roots α of $B\}$. Now for any positive root α occurring in the Lie algebra of B, we consider the corresponding copy of SL_2 in G and its Borel subgroup B_1. The weight space of $w\nu$ is B_1-invariant for otherwise ([9; 31.1]) there would be a weight space with weight $w\nu + i\alpha$, $i > 0$, and such a translate of $w\nu$ will have larger length which will be a contradiction to the extremalness of $w\nu$. Thus the dominant extremal weight ν_0 occurs in the B-socle of $H^0(G/B, \mathcal{L}(\lambda))$ which has weight $w_0\lambda$ by Lemma 2.2.9. Thus ν is a W-translate of this weight $w_0\lambda$. Also since the socle of $H^0(G/B, \mathcal{L}(\lambda))$

is one-dimensional we see that the weight space for any extremal weight of $H_{w_0}(\lambda)$ is one-dimensional.

The line bundle $\mathcal{L}(\lambda)$ is base point free. Therefore the restriction map on to sections over a T-fixed point $w.B/B$ is surjective for every $w \in W$. The torus T acts by the character $w\lambda$ on the fibre of this fixed point. This gives a geometric description of the one-dimensional extremal weight space $H^0(G/B, \mathcal{L}(\lambda))_{w\lambda}$, namely it is spanned by "the" T-semi-invariant section of $\mathcal{L}(\lambda)$ whose restriction to the fibre $\mathcal{L}(\lambda)|_{wB/B}$ is non-zero. This section vanishes at zB/B for $z \in W$ with $z\lambda \neq w\lambda$. Note that in $H^0(X_w, \mathcal{L}(\lambda))$ the restricted section is even B-semi-invariant so that its zero set is a union of the Schubert varieties X_z with $z \leq w$ and $z\lambda \neq w\lambda$.

To see the general case we note that the natural restriction map from $H_{w_0}(\lambda)$ to $H_w(\lambda)$ for $w \in W$ preserves the length of a weight and is surjective by Ramanathan (Proposition A.2.6). Therefore we see that the weight $w\lambda$ of $H_w(\lambda)$ is extremal and any other extremal weight of $H_w(\lambda)$ is also an extremal weight of $H^0(G/B, \mathcal{L}(\lambda))$. Now let us be given an extremal weight μ of $H_w(\lambda)$ and a non-zero section f of weight μ over G/B. Choose w' minimal in the Bruhat order so that $w'\lambda = \mu$. We claim $w' \leq w$. Otherwise the Mayer–Vietoris Lemma 2.2.11 gives a $g \in H^0(X_{w'} \cup X_w, \mathcal{L}(\lambda))_\mu$ with the same restriction to $w'.B/B$ as f, but vanishing on X_w. By Ramanathan Proposition A.2.6 this section g lifts to $H^0(G/B, \mathcal{L}(\lambda))_\mu$ and thus agrees with f, which is absurd. Here we have been using several times that T is semi-simple, so that if $M \to N$ is a surjective T-module map, $M_\mu \to N_\mu$ is surjective for every weight μ of N. $\square$

Remark 2.2.16 One can also prove the above proposition by induction on the length of w, using Corollary 2.2.10 and Proposition 2.2.5.

2.3 Dual Joseph Modules

For any character $\mu \in X(T)$, there exists an element $w \in W$, the Weyl group of G, such that $\mu_1 = w\mu \in X(T)^-$. We define $P(\mu) = H^0(X_{w^{-1}}, \mathcal{L}(w\mu))$. Thus the socle of $P(\mu)$ is of dimension one and has weight μ.

Lemma 2.3.1 *$P(\mu)$ is independent of w, i.e. for any $w_1, w_2 \in W$ with*

$w_1\mu = w_2\mu \in X(T)^-$, we have $H^0(X_{w_1^{-1}}, \mathcal{L}(w_1\mu)) = H^0(X_{w_2^{-1}}, \mathcal{L}(w_2\mu))$.

Proof: We denote by λ the translate of μ under W such that $\lambda \in X(T)^-$. Then recall ([9; 1.8, 1.10, 1.12]) that there exist elements $w_{\min}$ and $w_{\max}$ of the Weyl group W with the property that for any other w with $w\mu = \lambda$, we have $w_{\min} \leq w \leq w_{\max}$. Now consider the natural restriction map

$$H^0(X_{w_{\max}^{-1}}, \mathcal{L}(\lambda)) \longrightarrow H^0(X_{w^{-1}}, \mathcal{L}(\lambda)).$$

Since this map restricts to identity on the socles of the two modules, socles of both modules are one-dimensional and have weight μ, it is injective, and it is surjective according to Proposition A.2.6. Thus it is an isomorphism. This proves the proposition. $\square$

Definition 2.3.2 A B-module M is called *dual Joseph module* if M is isomorphic with $P(\mu)$ for some character μ.

Examples 2.3.3 1. For $\mu \in X(T)^-$ we have $P(\mu) = k_\mu$, the one-dimensional B-module with character μ.

2. For $\mu \in X(T)^+$ we have $P(\mu) = H^0(G/B, \mathcal{L}(w_0\mu))$.

Definition 2.3.4 (i) If $S' \subset S$ are B-invariant closed subspaces of G/B and $\lambda \in X(T)^-$, we define a *relative Schubert module* $Q(S, S', \lambda)$ by: $Q(S, S', \lambda) = \ker(\mathrm{res} : H^0(S, \mathcal{L}(\lambda)) \to H^0(S', \mathcal{L}(\lambda)))$.

(ii) If X_w is a Schubert variety its "boundary" ∂X_w is defined as the union of all Schubert varieties that are strictly contained in X_w. Thus the boundary is the complement in X_w of the Bruhat cell BwB/B.

(iii) For any $\mu \in X(T)$, we define a *minimal relative Schubert module*, denoted by $Q(\mu)$ by:

$$Q(\mu) = \ker(\mathrm{res} : H^0(X_{w_{\min}^{-1}}, \mathcal{L}(\lambda)) \to H^0(\partial X_{w_{\min}^{-1}}, \mathcal{L}(\lambda)))$$

where as before, $\lambda = w_{\min}\mu \in X(T)^-$ and $w_{\min}$ is a minimal such element in W.

Remark 2.3.5 Note that $Q(\mu) \hookrightarrow P(\mu)$. Also, the geometric description of the extremal weights of $P(\mu)$ tells us that an extremal weight of $P(\mu)$ other than μ does not restrict to zero on the boundary. Therefore μ is the only extremal weight of $Q(\mu)$.

Definition 2.3.6 A B-module M is said to have an *excellent filtration* if and only if there exists a filtration $0 \subset F_0 \subset F_1 \subset \cdots$ by B-modules such that $\bigcup F_i = M$ and $F_i/F_{i-1} \approx \oplus P(\lambda_i)$ for some $\lambda_i \in X(T)$. Here $\oplus$ stands for any number of copies, ranging from zero copies to infinitely many.

Remark 2.3.7 The property of having excellent filtration is closed under extension for finite dimensional B-modules. Thus for any short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of finite dimensional B-modules, M has excellent filtration whenever M_1 and M_2 both have excellent filtration.

In the next chapter, using the cohomological criterion for excellent filtrations, we will remove the finite dimensionality condition (cf. Corollary 3.2.10).

Definition 2.3.8 A B-module M is said to have a *relative Schubert filtration* if and only if there exists a filtration $0 \subset F_0 \subset F_1 \subset \cdots$ by B-modules such that $\bigcup F_i = M$ and $F_i/F_{i-1} \approx \oplus Q(\lambda_i)$ for some $\lambda_i \in X(T)$.

Remark 2.3.9 The property of having relative Schubert filtration is also closed under extension for finite dimensional B-modules.

In the next chapter we use Polo's theorem to give a criterion for B-modules to have an excellent filtration. Here we will now give examples of modules with relative Schubert filtration.

Lemma 2.3.10 *The relative Schubert module $Q(S, S', \lambda)$ has relative Schubert filtration for all B-invariant closed subsets $S' \subset S$ and any anti-dominant character λ.*

Proof: The proof is by induction on the number of Schubert varieties contained in S but not in S'.

First assume there is just one such Schubert variety, say X_w. Then $X_w \cap S' = \partial X_w$ and from the Mayer–Vietoris Lemma 2.2.11 one gets $Q(S, S', \lambda) = Q(X_w, \partial X_w, \lambda)$, which is either zero or $Q(w\lambda)$.

If there are more, choose a B-invariant S'' strictly between S and S' and consider the following exact sequence.

$$0 \longrightarrow Q(S, S'') \longrightarrow Q(S, S') \longrightarrow Q(S'', S') \longrightarrow 0.$$

Note that the exactness of this sequence is due to the Mayer–Vietoris Lemma 2.2.11.

By the induction hypothesis both the quotient and the submodule of $Q(S, S')$ have relative Schubert filtration. Now the Remark 2.3.9 proves the result. □

Another set of examples of modules with relative Schubert filtration is given by the following proposition.

Proposition 2.3.11 *For any B-invariant closed subset S of G/B and $\lambda \in X(T)^-$, $H^0(S, \mathcal{L}(\lambda))$ has a relative Schubert filtration with layers $Q(w\lambda)$. Moreover $Q(w\lambda)$ occurs only when $w\lambda$ is an extremal weight of $H^0(S, \mathcal{L}(\lambda))$, and has multiplicity one.*

Proof: The previous proof applies also for empty S' and the rest should be clear from the discussion. □

Corollary 2.3.12 *The modules $H_w(\lambda)$ has relative Schubert filtration for all $w \in W$ and $\lambda \in X(T)$.* □

Chapter 3

Polo's Theorem

In characteristic zero, the representations of reductive algebraic groups are completely reducible. This means that the irreducible representations are injective, as any extension of an irreducible by an irreducible is split exact. The dual Joseph modules introduced in the last chapter are not injective in the category of B-modules. Due to this non-injectivity the excellent filtrations are non-trivial filtrations of B-modules. However, in this chapter, we prove certain injectivity theorems for $P(\lambda)$.

In the first section, we prove Polo's theorem which says that the dual Joseph module $P(\lambda)$ is injective in a smaller category $\mathcal{C}_{\leq l(\lambda)}$.

In the second section, using a strong version of Polo's theorem, we give a cohomological criterion for a B-module to have an excellent filtration.

3.1 Polo's Theorem

We choose as in Bourbaki a linear functional *height* on $X(T) \otimes \mathbb{R}$ which is positive on all roots of B and injective on the lattice $X(T)$. We say that λ precedes μ in *length–height order* if either $l(\lambda) < l(\mu)$ or $[l(\lambda) = l(\mu)$ and the height functional takes a higher value on μ than on $\lambda]$. This defines a total order on $X(T)$—somewhat arbitrarily because of the freedom in the choice of the height functional—which captures the "highest weight category structure" corresponding with the dual Joseph modules. Rather than explaining what this means we ask the reader to look how the length–height order functions in proofs. For $\lambda \in X(T)$ we define $\mathcal{C}_{<\lambda}$ to consist of the B-modules all of whose weights strictly precede λ in length–height order. If M is a B-module then $M_{<\lambda}$ is the largest B-submodule of M that is in $\mathcal{C}_{<\lambda}$. Similarly one defines $\mathcal{C}_{\leq\lambda}$ and $M_{\leq\lambda}$. (For graded B-modules we will give a slightly different meaning to these notations.)

If $R \geq 0$ then $\mathcal{C}_{\leq R}$ ($\mathcal{C}_{<R}$) denotes the full subcategory of $\mathcal{C}_B$ whose objects are the modules all of whose weights have length not more than R (strictly less than R).

In this section we prove that the dual Joseph module $P(\lambda)$ is injective in $\mathcal{C}_{\leq l(\lambda)}$.

Lemma 3.1.1 *The category $\mathcal{C}_B$ of B-modules has enough injectives.*

Proof: Recall that for any subgroup H of a group G, the restriction functor res_H^G is exact. Further by the Frobenius reciprocity the induction functor ind_H^G is its right adjoint functor (see Proposition 2.1.6). The induction functor thus sends injective H-modules to injective G-modules. This makes $k[B]$, the ring of regular functions on B, an injective B-module as $k[B] = \mathrm{ind}_{\{e\}}^B(k)$, where the $\{e\}$ denotes the identity subgroup of B. Similarly, if M is a B-module, then $\mathrm{ind}_{\{e\}}^B \mathrm{res}_{\{e\}}^B M$ is injective, and it contains M as a submodule (exercise). Therefore $\mathcal{C}_B$ has enough injectives. □

Remark 3.1.2 A useful property of injectives in $\mathcal{C}_B$ is that if one tensors them with any B-module, the result is again injective ([11; I 3.10]).

Corollary 3.1.3 *The subcategories $\mathcal{C}_{\leq R}$, $\mathcal{C}_{<R}$, $\mathcal{C}_{\leq\lambda}$, $\mathcal{C}_{<\lambda}$ have enough injectives.*

Proof: We prove the corollary for $\mathcal{C}_{\leq R}$. The proof is similar for the other cases.

We denote by $M_{\leq R}$ the largest B-submodule of a B-module M whose weights have length less than or equal to R. Then $M \mapsto M_{\leq R}$ is the right adjoint of the embedding functor $\mathcal{C}_{\leq R} \to \mathcal{C}_B$, which is exact. So if M is an injective B-module, then $M_{\leq R}$ is injective in the category $\mathcal{C}_{\leq R}$. □

Remark 3.1.4 Beware that $M_{\leq R}$ is usually much smaller than the largest T-submodule of M whose weights have length less than or equal to R. The latter would be simply the sum of those weight spaces whose weight has length less than or equal to R.

Remark 3.1.5 The description of the extremal weights of $H_w(\lambda)$ says that $H_w(\lambda) \in \mathcal{C}_{\leq R}$ for $k_\lambda \in \mathcal{C}_{\leq R}$. Therefore for μ with $l(\mu) \leq l(\lambda) \leq R$, the module $P(\mu)$ (and hence $Q(\mu)$) is an object of $\mathcal{C}_{\leq R}$.

For a module M to be injective in a category $\mathcal{C}$ we need to have vanishing of the Ext functors for M ([23; Ch III]). Before trying to prove such vanishing for a dual Joseph module we first make some remarks.

Remark 3.1.6 Note that given a B-module N one may write it as a filtered union $N = \lim_j N_j$ of finite dimensional submodules N_j. This construction also has the property that the standard injective resolutions [11; Hochschild complex] of the N_j converge to an injective resolution of N. Thus to prove $\mathrm{Ext}^i(M_0, ?) = 0$ for a fixed finite dimensional $M_0 \in \mathcal{C}_B$ and fixed i, we need only prove $\mathrm{Ext}^i(M_0, N) = 0$ for a *finite dimensional* N.

Remark 3.1.7 Further given a finite dimensional B-module N, using Borel's Fixed Point Theorem we get a one-dimensional B-module k_ν with weight ν such that $0 \to k_\nu \to N \to Q \to 0$.

Writing its associated long exact sequence of Ext^i groups, we see that $\mathrm{Ext}^i(M_0, N) = 0$ whenever $\mathrm{Ext}^i(M_0, Q) = \mathrm{Ext}^i(M_0, k_\nu) = 0$.

Therefore a B-module M_0 with $\mathrm{Ext}^i(M_0, k_\nu) = 0$ for all ν, is injective.

Remark 3.1.8 Let $\mathcal{C}$ be a category with sufficiently many injectives and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$ with the following property: whenever $M_1, M_2 \in \mathcal{C}'$, then for every exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathcal{C}$, M also lies in $\mathcal{C}'$. Then for M and N in $\mathcal{C}'$, we have $\mathrm{Ext}^1_{\mathcal{C}}(M, N) = \mathrm{Ext}^1_{\mathcal{C}'}(M, N)$ (cf. [23; Ch III §1, §8]). This observation is useful in the case of $C' = \mathcal{C}_{\leq R}$ and $C = \mathcal{C}_B$. Since T-modules are semi-simple, every exact sequence of B-modules $0 \to M_1 \to M \to M_2 \to 0$ splits as T-modules and thus if $M_1, M_2 \in \mathcal{C}_{\leq R}$ then $M \in \mathcal{C}_{\leq R}$ indeed.

Definition 3.1.9 The *injective hull* of a B-module M, is an injective B-module containing M whose socle is $\mathrm{soc}(M)$. It is unique up to non-canonical isomorphism.

Theorem 3.1.10 (Polo's theorem) *Let $\lambda \in X(T)^-$ with length $l(\lambda)$. Then*

$$H_w(\lambda) = H^0(X_w, \mathcal{L}(\lambda))$$

is the injective hull of $k_{w\lambda}$ in $\mathcal{C}_{\leq l(\lambda)}$.

Proof: (After H.H. Andersen.) The dual Joseph module $H_w(\lambda)$ has one-dimensional socle with weight $w\lambda$. Thus it is enough to prove that $H_w(\lambda)$ is injective in $\mathcal{C}_{\leq l(\lambda)}$.

By a familiar application of Zorn's Lemma—cf. proof of Prop. 7.2 in [23; Ch. III]—it suffices to prove that $H_w(\lambda)$ is injective in the full subcategory of $\mathcal{C}_{\leq l(\lambda)}$ consisting of finite dimensional modules. Also note that for finite dimensional M (see [11; I Ch. 4])

$$\begin{aligned}\operatorname{Ext}^i_B(M, H_w(\lambda)) &= H^i(B, H_w(\lambda) \otimes M^*) \\ &= \operatorname{Ext}^i_B(H_w(\lambda)^*, M^*).\end{aligned}$$

Therefore it is enough to prove that $\operatorname{Ext}^1_{B,\lambda}(H_w(\lambda)^*, M) = 0$, where $\operatorname{Ext}^1_{B,\lambda}$ denotes the first derived functor of the functor Hom in the category $\mathcal{C}_{\leq l(\lambda)}$. We will prove using induction on *length of* w that $\operatorname{Ext}^1_{B,\lambda}(H_w(\lambda)^*, k_\nu) = 0$.

When $w = e$, we have $H_w(\lambda) = k_\lambda$. Also $\operatorname{Hom}_B(k_{-\lambda}, M) = M^U_{-\lambda}$, the U-invariants in $M_{-\lambda}$. But in $\mathcal{C}_{\leq l(\lambda)}$ we have $M^U_{-\lambda} = M_{-\lambda}$ because $\lambda \in X(T)^-$ (exercise, cf. proof of 2.2.15). Thus the Hom functor is identified with the functor $M \mapsto M_{-\lambda}$. This functor is exact. Therefore $\operatorname{Ext}^1_{B,\lambda}(k_{-\lambda}, M) = 0$.

Let $H_w(\lambda)$ be the injective hull of k_λ. Let $s \in W$ be a simple reflection such that $l(sw) = l(w) + 1$. To complete the inductive argument, we need to prove that $H_{sw}(\lambda)$ is injective in $\mathcal{C}_{\leq l(\lambda)}$.

Recall that by Proposition 2.2.5

$$H_{sw} = H_s \circ H_w = \operatorname{ind}_B^{P_s} \circ H_w.$$

Further, by using the Frobenius reciprocity repeatedly, we obtain:

$$\begin{aligned}\operatorname{Hom}_B(H_s(H_w(M))^*, N) &= \operatorname{Hom}_{P_s}(H_s(H_w(M))^*, H_s(N)) \\ &= \operatorname{Hom}_{P_s}(H_s(N)^*, H_s(H_w(M))) \\ &= \operatorname{Hom}_B(H_w(M)^*, H_s(N)).\end{aligned}$$

Thus we get that

$$(*) \qquad \operatorname{Hom}_B(H_{sw}(M)^*, N) = \operatorname{Hom}_B(H_w(M)^*, H_s(N))$$

This proves that the functor $\operatorname{Hom}_B(H_{sw}(\lambda)^*, ?)$ is the composition of the two functors $H_s : \mathcal{C}_B \to \mathcal{C}_B$ and $\operatorname{Hom}_B(H_w(\lambda)^*, ?)$. Now recall the Grothendieck spectral sequence ([11]) for two functors $F : \mathcal{C} \to \mathcal{C}'$ and $F' : \mathcal{C}' \to \mathcal{C}''$ with F, F' left exact and F mapping injective objects in $\mathcal{C}$ to objects acyclic for F'. It says that

$$(R^n F')(R^m F)(M) \Rightarrow R^{n+m}(F' \circ F)(M) \quad \forall M \in \mathcal{C}.$$

In particular, if M is acyclic for F, *i.e.* if $(R^m F)M = 0$ for $m > 0$, then the spectral sequence degenerates to

$$(R^n F')F(M) = R^n(F' \circ F)(M).$$

The latter is all we will use about the Grothendieck spectral sequence and it can of course be proved directly—without spectral sequences—by induction on n, using the long exact sequences associated with the exact sequence

$$0 \longrightarrow M \longrightarrow Q_M \longrightarrow Q_M/M \longrightarrow 0,$$

where Q_M is the injective hull of M.

We want to use all this for $F = H_s : \mathcal{C}_B \to \mathcal{C}_B$ and $F' = \mathrm{Hom}_B(H_w(\lambda)^*, ?)$. We have to check the conditions. To this end we need

Lemma 3.1.11 *Let M be a B-module which is a quotient of a P_s-module. Then M is $\mathrm{ind}_B^{P_s}$-acyclic. In particular, $H_w(\lambda)$ is $\mathrm{ind}_B^{P_s}$-acyclic.*

Proof: Note that the restriction map $H^0(G/B, \mathcal{L}(\lambda)) \to H_w(\lambda)$ is surjective by Ramanathan (cf. Proposition A.2.6), so that $H_w(\lambda)$ is indeed a quotient of a P_s-module. Now P_s/B is a projective line $\mathbb{P}^1$, so there is no higher cohomology than in degree 1, and if M is a quotient of the P_s-module N then $R^1 \mathrm{ind}_B^{P_s}(M) = H^1(P_s/B, \mathcal{L}(M))$ is a quotient of $R^1 \mathrm{ind}_B^{P_s}(N)$, which vanishes because $\mathcal{L}(N)$ is a trivial bundle (see also [7]). $\square$

Now for the spectral sequence to apply we must check the vanishing of $\mathrm{Ext}_B^m(H_w(\lambda)^*, H_s(N)) = H^m(B, H_w(\lambda) \otimes H_s(N))$ for $m > 0$, when N is an injective B-module. But then $H_s(N) = \mathrm{ind}_B^{P_s}(N)$ is an injective P_s-module, and if M is a finite dimensional B-module, $\mathrm{Ext}_B^m(H_s(M)^*, H_s(N)) = \mathrm{Ext}_{P_s}^m(H_s(M)^*, H_s(N))$ by 2.1.10, so this vanishes and $H^m(B, H_s(M) \otimes H_s(N))$ thus vanishes for any B-module M. This means (use Remark 3.1.2) that at least we have a spectral sequence for the functors F and F'', with $F'' = H^0(B, ? \otimes H_s(N))$. The composite functor $F'' \circ F$ is just $H^0(B, ? \otimes H_s(N))$, by Frobenius reciprocity and 2.1.10. The lemma gives us that $H^m(B, H_w(\lambda) \otimes H_s(N)) = R^m(F'' \circ F)(H_w(\lambda)) = R^m(F'') \circ F(H_w(\lambda)) = 0$ for $m > 0$, as required.

We may thus state that

$$\mathrm{Ext}^i_B(H_w(\lambda)^*, R^j H_s(k_\nu)) \Rightarrow \mathrm{Ext}^{i+j}_B(H_{sw}(\lambda)^*, k_\nu)$$

and finish the proof as follows.

1. The case when ν is anti-dominant with respect to s, *i.e.* $H_s(\nu) \neq 0$.
In this case using Kempf's vanishing theorem we see that k_ν is acyclic for H_s. Therefore our spectral sequence degenerates and gives:

$$\mathrm{Ext}^i_B(H_{sw}(\lambda)^*, k_\nu) = \mathrm{Ext}^i_B(H_w(\lambda)^*, H_s(k_\nu))$$

Now we use the inductive hypothesis to get the required result.

2. ν is not anti-dominant with respect to s.
We put $\mu = s(\nu - \rho)$, where ρ is the half sum of the roots of B. Then μ is anti-dominant with reference to s and moreover k_ν is the socle of $\rho \otimes H_s(\mu)$. Also we have: $R^j H_s(\rho \otimes H_s(\mu)) = R^j H_s(\rho) \otimes H_s(\mu)$ by the tensor identity ([11]). But $R^j H_s(\rho) = 0 \quad \forall j \geq 0$ (cohomology of line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1$, cf. [11; II 5.2]).
Thus we have $\mathrm{Ext}^i(H_w(\lambda), R^j H_s(\rho \otimes H_s(\mu))) = 0$ for all i and j. Now consider

$$0 \longrightarrow k_\nu \longrightarrow \rho \otimes H_s(\mu) \longrightarrow Q \longrightarrow 0.$$

Writing down part of the associated long exact sequence of B-cohomology gives $\mathrm{Hom}_B(H_{sw}(\lambda)^*, Q) \rightarrow \mathrm{Ext}^1_B(H_{sw}(\lambda)^*, k_\nu) \rightarrow 0$. But one can check (cf. [11; II 5.2]) that all weights of Q are strictly less in length than ν. As the socle of $H_{sw}(\lambda)$ has a weight at least as long as ν, one must have $\mathrm{Hom}_B(Q^*, H_{sw}(\lambda)) = 0$. This gives the vanishing of the Ext. □

Lemma 3.1.12 *Let M be a B-module with an excellent filtration. Then M is $\mathrm{ind}_B^{P_s}$-acyclic.*

Proof: Use Remark 2.1.5 and Lemma 3.1.11 to prove this lemma. □

3.2 Cohomological Criterion

In this section we give a criterion for a B-module to have an excellent filtration. First we prove a stronger version of Polo's theorem.

Remark 3.2.1 Polo's theorem and Remark 3.1.8 preceding it says that: $\mathrm{Ext}^1_B(H_w(\lambda)^*, M) = 0$, where $w \in W$ and $\lambda \in X(T)^-$ and $M \in \mathcal{C}_{\leq \lambda}$.

The following theorem proves that this equality is true in case of the higher derived functors too.

Theorem 3.2.2 (Strong form of Polo's Theorem) *Let $\lambda \in X(T)^-$ and $M \in \mathcal{C}_{\leq l(\lambda)}$. Then, for $w \in W$, $i > 0$,*

$$\mathrm{Ext}^i_B(H_w(\lambda)^*, M) = 0.$$

Proof: We merely extend H.H. Andersen's proof of Polo's theorem (Theorem 3.1.10) to prove this extension. We go through the old proof. This time we want to prove $\mathrm{Ext}^i(H_w(\lambda)^*, k_\nu) = 0$ for $i > 0$ and $k_\nu \in \mathcal{C}_{\leq l(\lambda)}$.

When $w = e$, the identity element of the Weyl group, we take the minimal injective resolution $I^*(\lambda)$ of k_λ in $\mathcal{C}_B$, as in [11; II 4.8–9]. We claim that all the weights other than λ occurring in $I^1(\lambda)$ are necessarily *longer* than λ. Indeed $I^1(\lambda) = k_\lambda \otimes k[U]$ and λ has non-negative inner product with every non-zero weight of $k[U]$ because λ is anti-dominant. For higher values of i the weights of $I^i(\lambda)$ are in the same region (see [11; II 4.8–9]) and are thus also strictly longer than λ. Therefore $\mathrm{Ext}^i(k_{-\lambda}, k_\nu) = \mathrm{Ext}^i(k_{-\nu}, k_\lambda) = 0$, which proves the case when w has length zero.

The rest of the proof of Theorem 3.1.10 extends without trouble to give this stronger version. At the end, where the weights of Q are all strictly shorter than ν, use that we may assume by induction on the length of weights that all $\mathrm{Ext}^i(H_{sw}(\lambda)^*, Q)$ vanish. □

Exercise 3.2.3 Complete the above proof by filling in all the details.

Let M be a finite dimensional B-module. Then, $\mathrm{Ext}^i(M, N) = H^i(B, M^* \otimes N)$. Thus the injectivity of $H_w(\lambda)$ can be interpreted in terms of B-acyclicity. (Recall that a B-module M is B-acyclic if $H^i(B, M) = 0$ for $i > 0$.)

Corollary 3.2.4 *For $\lambda, \mu \in X(T)$, $P(\lambda) \otimes P(\mu)$ is B-acyclic.*

Proof: Let $(\mu, \mu) \leq (\lambda, \lambda)$, Then we have:

$$H^i(B, P(\lambda) \otimes P(\mu)) = \mathrm{Ext}^i_B(P(\lambda)^*, P(\mu))$$

Now the strong Polo's theorem gives the result. □

Recall that a B-module M is said to have an *excellent filtration* if there exists a filtration $0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots$ by B-modules such that $\bigcup F_i = M$ and $F_i/F_{i-1} \approx \oplus P(\lambda_i)$ for some $\lambda_i \in X(T)$.

Corollary 3.2.5 *The tensor product of two modules with excellent filtrations is B-acyclic.* □

Theorem 3.2.6 *For $\lambda, \mu \in X(T)$, $P(\lambda) \otimes Q(\mu)$ is B-acyclic.*

Proof: If $l(\mu) \leq l(\lambda)$ then $Q(\mu) \in \mathcal{C}_{\leq l(\lambda)}$ and thus $H^i(B, P(\lambda) \otimes Q(\mu)) = 0$ for $i > 0$.

If $l(\mu) > l(\lambda)$ then we let w_μ denote the minimal element of the Weyl group which takes μ to the anti-dominant chamber. We will prove the theorem by induction on the length of w_μ.

When $l(w_\mu) = 0$, we have $\mu \in X(T)^-$ and therefore $Q(\mu) = P(\mu)$ and the result follows.

When $l(w_\mu) > 0$, we look at the short exact sequence

$$0 \longrightarrow Q(\mu) \longrightarrow P(\mu) \longrightarrow H^0(\partial X_{w_\mu^{-1}}, \mathcal{L}(w_\mu \mu)) \longrightarrow 0.$$

The quotient has a filtration whose associated graded consists of direct sums of relative Schubert modules $Q(\tau)$ with $l(w_\tau) < l(w_\mu)$ and thus we can use an induction hypothesis for the quotient. Now the associated long exact sequence of Ext gives the result. □

We now prove that a weaker condition than the one suggested by Theorem 3.2.6 is sufficient for a module to have an excellent filtration.

Theorem 3.2.7 (Cohomological criterion for excellent filtration) *Let M be a B-module such that for every $\lambda \in X(T)$, $H^1(B, M \otimes Q(\lambda)) = 0$. Then, M has excellent filtration.*

Proof: First, we order the characters in the length–height order. Let $\lambda_1, \ldots$ be our enumeration of $X(T)$ according to length–height order. Let $\{F_i\}$ be the length–height filtration of M, *i.e.* $F_i = M_{\leq \lambda_i}$ is the largest B-submodule whose weights are in $\{\lambda_1, \lambda_2, \ldots, \lambda_i\}$.

We will prove that the length–height filtration of M is an excellent filtration of M. In fact we will show that $F_i/F_{i-1} \approx \oplus P(\lambda_i)$ for $i \geq 0$. If not, take i to be minimal so that it fails.

Consider the short exact sequence: $\mathcal{E} : 0 \to F_{i-1} \to M \to R \to 0$. All the weights occurring in $\mathrm{soc}(R)$ are strictly larger than λ_{i-1} in the length–height order. Now for a character η such that $l(\eta) \leq l(\lambda_{i-1})$, we write the long exact sequence of B-cohomology associated to $\mathcal{E} \otimes Q(\eta)$. We get because of the Acyclicity Theorem 3.2.6 that $H^1(B, R \otimes Q(\eta)) = 0$. Therefore we do not cheat if we replace M by R in the sequel. The effect of this is that we may further assume $F_{i-1} = 0$, so that the socle of F_i is entirely of weight λ_i. Let us show next that $H^1(B, F_i \otimes k_\eta) = 0$. There are two cases. The first case is that the height of $-\eta$ is at least that of λ_i. Then all weights of $N := F_i \otimes k_\eta$ are of negative or zero height, as the socle of N is of weight $\lambda_i - \eta$. But then $\mathrm{Ext}_B(k, N)$ clearly vanishes, cf. [11; II 4.10].

The second case is that $-\eta$ precedes λ_i in length–height order, so that $\mathrm{Hom}(Q(\eta)^*, M/F_i) = 0$. It follows that $\mathrm{Ext}^1(Q(\eta)^*, F_i) = 0$. Further, looking at

$$(*) \qquad 0 \longrightarrow \eta \longrightarrow Q(\eta) \longrightarrow \frac{Q(\eta)}{\eta} \longrightarrow 0,$$

with η short for k_η, we get $\mathrm{Hom}_B\big((Q(\eta)/\eta)^*, F_i\big) \to \mathrm{Ext}^1(k_{-\eta}, F_i) \to 0$. However $Q(\eta)/\eta$ has weights which are strictly less in length than λ_i. Therefore we have $\mathrm{Hom}_B((\frac{Q(\eta)}{\eta})^*, F_i) = 0$ and the second case follows too. Thus F_i is injective in $\mathcal{C}_{\leq l(\lambda_i)}$ with a socle purely of weight λ_i. This proves that F_i is a direct sum of copies of $P(\lambda_i)$, with as many copies as the dimension of the socle of F_i. $\square$

From the proof we actually get:

Corollary 3.2.8 *For a B-module with excellent filtration the length–height filtration is an excellent filtration.* $\square$

This corollary is important for checking that the length–height order leads to a highest weight category structure in the sense of Cline-Parshall-Scott. We will not get into that and just tell everything in terms of the length–height order itself.

Corollary 3.2.9 *An injective B-module has an excellent filtration.* □

Corollary 3.2.10 *The property of excellent filtration is closed under extension.*

Proof: Let M_1 and M_2 be two B-modules (maybe infinite dimensional) with excellent filtration. Let M be a B-module such that we have an exact sequence $0 \to M_1 \to M \to M_2 \to 0$. Tensor this exact sequence by $Q(\mu)$ and write its associated long exact sequence of B-cohomology and use the cohomological criterion. □

Lemma 3.2.11 *Let M be a B-module with excellent filtration and let w be an element of the Weyl group. Then the module $H_w(M)$ has excellent filtration.*

Proof: We fix a set of generators $S = \{s_1, \ldots, s_l\}$ of W such that each of its elements is a simple reflection. Let $w = s_1 \cdots s_n$ be a reduced expression of w. By Proposition 2.2.5, we have $H_w(M) = H_{s_1} \circ \cdots \circ H_{s_n}(M)$. Therefore it is enough to prove that $H_s(M)$ has excellent filtration for every simple reflection $s \in S$. We first consider the case when $M = P(\mu)$ for some character μ. Let $\mu_1 = w_\mu^{-1}\mu$ denote the anti-dominant weight in its Weyl group orbit. Then $P(\mu) = H_{w_\mu}(\mu_1)$ and $H_s(P(\mu))$ is by Proposition 2.2.5 either isomorphic to $H_{w_\mu}(\mu_1) = P(\mu)$ or to $H_{sw_\mu}(\mu_1) = P(s\mu)$. Therefore we have proved the claim for $M = P(\mu)$.

Now we will use induction to prove the claim for all of M.

Let $0 \subset F_1 \subset F_2 \subset \cdots$ be an excellent filtration of M. Note that F_1 is a direct sum of copies of $P(\mu)$ for some μ. Therefore we know that $H_s(F_1)$ has excellent filtration. Let m be the (hypothetical) least integer such that $H_s(F_m)$ has excellent filtration but $H_s(F_{m+1})$ does not have excellent filtration. Consider the exact sequence

$$0 \longrightarrow F_m \longrightarrow F_{m+1} \longrightarrow M_1 \longrightarrow 0.$$

The module M_1 is isomorphic to a direct sum of copies of $P(\nu)$ for some character ν. This exact sequence gives rise to the exact sequence

$$0 \longrightarrow H_s(F_m) \longrightarrow H_s(F_{m+1}) \longrightarrow H_s(M_1) \longrightarrow 0.$$

The surjectivity of this exact sequence is due to the $\mathrm{ind}_B^{P_s}$-acyclicity of F_m (Lemma 3.1.12). Now the cohomological criterion for excellent filtration—or common sense if the modules are finite dimensional—gives us the result. □

3.3 Relative Schubert Modules

In this section we state and prove (in the form of exercises) analogous statements for the relative Schubert modules.

Definition 3.3.1 Let $\mathcal{C}_\lambda$ denote the full subcategory of $\mathcal{C}_B$ whose objects are the modules M such that if μ is a weight of M then either $\mu = \lambda$ or $l(\mu) < l(\lambda)$.

Note that $\mathcal{C}_{<l(\lambda)} \subsetneq \mathcal{C}_\lambda \subsetneq \mathcal{C}_{\leq l(\lambda)}$ if $\lambda \neq 0$.

Exercise 3.3.2 Prove that $Q(\lambda)$ is injective in $\mathcal{C}_\lambda$.

Hint: Use the injectiveness of $P(\lambda)$ in $\mathcal{C}_{\leq l(\lambda)}$ and the proof of the Corollary 3.1.3.

The proof of the cohomological criterion for excellent filtration extends easily to give us the following result.

Exercise 3.3.3 (The cohomological criterion for relative Schubert filtration). Prove that a B-module M has relative Schubert filtration if and only if $H^1(B, M \otimes P(\mu)) = 0$ for all characters μ.

Hint: This time order the weights a little differently, using the negative of the height function.

Chapter 4

Donkin's Conjecture

Let k now be an algebraically closed field of *positive characteristic* p , and let G be our connected reductive group over k. Let M be G-module. A filtration $\mathcal{F}$ of M is called *good* if the successive quotients are isomorphic to a direct sum of copies of $P(\mu)$ with $\mu \in X(T)^+$. In this chapter we prove Donkin's conjecture for good filtrations. The best known half of this conjecture is the (older) conjecture stating that for $\lambda,\ \mu \in X(T)^+,\ P(\lambda) \otimes P(\mu)$ has good filtration. The crucial idea (due to O. Mathieu) is to study the G-modules which are embedded in a graded B-algebra with a "canonical splitting".

In the first section we give the definition and basic properties of good filtration. We also give the relationship between the excellent filtrations and good filtrations.

In the second section we give a criterion for existence of a good filtration for a G-module. This criterion works in a very specialized case of a G-module embedded inside a graded B-algebra each of whose graded components has an excellent filtration and only one weight in its socle. However, as we will see in the last section, this criterion gives us the proof of Donkin's conjecture.

This criterion leads us to study what we call Frobenius-linear endomorphisms of a graded k-algebra R. A *splitting* σ of R is a Frobenius-linear endomorphism such that $\sigma(1) = 1$. The Frobenius splittings were introduced by Mehta and Ramanathan in [24]. Following Mathieu, we then introduce the notion of a canonical splitting of R and prove the crucial proposition that the image of a B-submodule of R under a canonical splitting is again a B-submodule.

The criterion for good filtration relates the concept of canonical splitting and that of good filtration. This gives the proof of Donkin's conjecture.

4.1 Good Filtrations

Definition 4.1.1 Let M be a G-module. A filtration $\mathcal{F} = F_0 \subset F_1 \subset \cdots$ of M by G-submodules is said to be a *good filtration* if

(i) $\bigcup_i F_i = M$.

(ii) $F_i/F_{i-1} \simeq \bigoplus P(\mu_i)$ with $\mu_i \in X(T)^+$.

The reader may have noticed the similarity between excellent filtration of a B-module and good filtration of a G-module. Indeed the questions of a B-module M having excellent filtration and $\mathrm{ind}_B^G(M)$ having good filtration are related. First we see what happens if $\mathrm{ind}_B^G(M) = M$.

Exercise 4.1.2 Let M be a G-module. Show that the length–height filtration of M is not just a filtration by B-submodules, but one by G-submodules. (Hint: Consider a minimal counterexample and factor out an irreducible G-submodule.)

Exercise 4.1.3 Let M be a G-module. Prove that the following are equivalent:

(i) M has a good filtration.

(ii) M has an excellent filtration. (That is, $\mathrm{res}_B^G(M)$ has one, but recall from 2.1.7 that we embed $\mathcal{C}_G$ in $\mathcal{C}_B$.)

(iii) The length–height filtration of M is a good filtration.

Remark 4.1.4 As the property of having excellent filtration is closed under extension, we see that the property of having a good filtration is also closed under extension.

We also have a cohomological criterion for good filtration which is analogous to the one for existence of an excellent filtration. (It is much older.)

Proposition 4.1.5 (Donkin) ([11; II 4.16]) *Let M be a G-module. Then M has a good filtration if and only if for every dual Weyl module $P(\lambda)$, $\lambda \in X(T)^+$, one has $H^1(G, M \otimes P(\lambda)) = 0$.* □

Corollary 4.1.6 *Let $M = M_1 \oplus M_2$ be a direct sum of two G-modules. Then M has good filtration if and only if both M_1 and M_2 have good filtration.* □

Exercise 4.1.7 Use Lemma 3.2.11 to show that if M has excellent filtration, $\mathrm{ind}_B^G(M)$ has good filtration.

4.2 Criterion for Good Filtrations

In this section we give a criterion for existence of good filtrations. Unlike the cohomological criterion, which is intrinsic, this criterion depends upon an embedding of the given G-module into a graded B-algebra. To motivate this approach we look at Donkin's conjecture.

Remark 4.2.1 Donkin's conjecture claims that for λ, μ two dominant characters the module $P(\lambda) \otimes P(\mu)$ has good filtration. Now geometrically $P(\lambda) \otimes P(\mu)$ can be interpreted as $P(\lambda) \otimes P(\mu) = H^0(G/B \times G/B, \mathcal{L})$ where $\mathcal{L}$ is the line bundle $\mathcal{L}(w_0\lambda) \times \mathcal{L}(w_0\mu)$ on $G/B \times G/B$. The variety $G \times^B G/B = G/B \times G/B$ contains $Bw_0B \times^B G/B$ as an open subset. Therefore the natural restriction map gives a natural embedding of $P(\lambda) \otimes P(\mu)$ into the graded B-algebra $\bigoplus_{j=0}^{\infty} H^0(Bw_0B \times^B G/B, \mathcal{L}^j)$. This B-algebra is induced from the T-module $\bigoplus_j H^0(Tw_0B \times^B G/B, \mathcal{L}^j)$ and is therefore injective. Hence by the cohomological criterion, it has excellent filtration.

Motivated by this remark, we state the following criterion for good filtration. First a definition.

Definition 4.2.2 Let $A = \oplus_i A^i$ be a graded B-algebra. We define a B-subalgebra $A_{\leq\lambda}$ of A by $A_{\leq\lambda} = \oplus_i A^i_{\leq i\lambda}$. (Recall that $M_{\leq\mu}$ is the largest B-submodule of M which is in the category $\mathcal{C}_{\leq\mu}$.)

Theorem 4.2.3 (p-root closure and good filtration) *Let $A = \bigoplus_i A^i$ be a graded B-algebra such that*

(i) $A^0 = k$.

(ii) *A has excellent filtration.*

(iii) *There exists $\lambda \in X(T)^+$ such that in* $\operatorname{soc}(A^j)$ *only $j \cdot \lambda$ occurs as weight.*

Let S be graded subalgebra which is a graded G-module and which is p-root closed (i.e. $a^p \in S \Rightarrow a \in S$). Then S has good filtration.

Proof: We wish to prove that each S^r has good filtration and we may restrict attention to a given r. We know by the cohomological criterion for excellent filtration (Theorem 3.2.7) that each of A^j has excellent filtration. Therefore for any m, the rescaled B-algebra $A_1 = \bigoplus_i A^{m \cdot i}$ with $A_1^i = A^{im}$ also has excellent filtration. Therefore we may assume that $S^1 \neq 0$.

The socle of S^i contains only a single weight $i\lambda$. Further as S is a G-module, $i\lambda$ is an extremal weight of S^i and all extremal weights are in the same Weyl group orbit as $i\lambda$.

Therefore we have $S \subset A_{\leq\lambda}$.

The length–height filtration of A is excellent. Further as the socle of A^j has no other weight than $j \cdot \lambda$, we see that the first non-zero module in this filtration of A^j is $(A^j)_{\leq j\lambda}$. Therefore $(A_{\leq\lambda})^j$ is isomorphic to $\bigoplus P(j \cdot \lambda)$.

To get a firm hold of the situation we need a technical sublemma that gives more insight in the algebra structure of $A_{\leq\lambda}$. That will allow us to pass to convenient subalgebras. The reader is advised to pass over this sublemma quickly.

Sublemma 4.2.4 *The graded B-algebra $A_{\leq\lambda}$ may be reconstructed from its "subalgebra of socles" $\bigoplus_j \operatorname{soc}_B(A^j)$. More generally, any graded subalgebra of $\bigoplus_j \operatorname{soc}_B(A^j)$ is the subalgebra of socles of a suitable graded subalgebra $\widetilde{A}$ of $A_{\leq\lambda}$, with $\widetilde{A}$ having excellent filtration.*

Proof: We have seen already that $(A_{\leq\lambda})^j$ is isomorphic as a B-module to a direct sum of copies of $P(j\lambda)$, with the number of copies equal to the dimension of $\operatorname{soc}((A_{\leq\lambda})^j)$. To say it more canonically—which one must, in view of the task at hand—there is a canonical isomorphism of B-modules

$$P(j\lambda) \otimes \operatorname{soc}((A_{\leq\lambda})^j) \otimes k_{-j\lambda} \longrightarrow (A_{\leq\lambda})^j.$$

So that is how we reconstruct $A_{\leq\lambda}$ as a B-module. To get the ring structure, note that multiplication is given by B-module maps

$$(A_{\leq\lambda})^r \otimes (A_{\leq\lambda})^s \longrightarrow (A_{\leq\lambda})^{r+s}.$$

Thus we are done with the first half of the lemma if we show that restriction defines an isomorphism from

$$\mathrm{Hom}_B((A_{\leq\lambda})^r \otimes (A_{\leq\lambda})^s, (A_{\leq\lambda})^{r+s})$$

to

$$\mathrm{Hom}_B(\mathrm{soc}((A_{\leq\lambda})^r) \otimes \mathrm{soc}((A_{\leq\lambda})^s), \mathrm{soc}((A_{\leq\lambda})^{r+s})).$$

For surjectivity one uses Polo's theorem with R equal to the length of $(r+s)\lambda$. To see injectivity, consider a map

$$f : (A_{\leq\lambda})^r \otimes (A_{\leq\lambda})^s \longrightarrow (A_{\leq\lambda})^{r+s}$$

in the kernel. If f is not zero, its image must hit the socle of $(A_{\leq\lambda})^{r+s}$. But then it must be non-zero on the weight space $((A_{\leq\lambda})^r \otimes (A_{\leq\lambda})^s)_{(r+s)\lambda}$. And that is just $\mathrm{soc}((A_{\leq\lambda})^r) \otimes \mathrm{soc}((A_{\leq\lambda})^s)$ as one sees by looking at lengths and heights. The rest of the sublemma follows similarly. $\square$

Encouraged by the sublemma we let $I(S)$ denote the graded subalgebra of $A_{\leq\lambda}$ whose j^{th} component is the injective hull of S^j in the category $\mathcal{C}_{\leq j\cdot\lambda}$. The subalgebra $I(S) \subset A_{\leq\lambda}$ clearly has excellent filtration. In fact $I(S)$ is a direct summand of $A_{\leq\lambda}$ and thus the filtration from its grading is an excellent filtration! Therefore we replace A by $I(S)$.

We will prove that $S = A$.

We can assume, by rescaling again if necessary, that $S^1 \neq I(S^1)$. Therefore there exists a copy of $P(\lambda) \subset I(S^1)$ such that S^1 does not contain $P(\lambda)$. We denote by A_1 the algebra generated by this $P(\lambda)$. Phrased differently, we let A_1 be the graded subalgebra with excellent filtration whose socle algebra is generated by the socle of our chosen copy of $P(\lambda)$. Let $A_2 = A_1 \cap S$. The G-algebra A_2 is again p-root closed in A_1. One may quickly dispense of the case that $A_1^j = 0$ for large j.

We choose a parabolic subgroup P such that λ extends as a character to P and P is maximal for this property. The line bundle $\mathcal{L} = \mathcal{L}(w_0 \cdot \lambda)$ is very ample on G/P ([11; II 8.5]). Further, we have $H^0(G/B, \mathcal{L}) = H^0(G/P, \mathcal{L})$.

Thus we can restrict our attention to the situation

$$S = A_2 \subset A_1 = A = \bigoplus_i H^0(G/P, \mathcal{L}^i).$$

Consider the projective space $\mathbb{P}(A^1)$ of one-dimensional *quotients* of A^1. We have a rational map $f : \mathbb{P}(A^1) \to \mathbb{P}(S^1)$. However, the image of G/P, under the canonical embedding, lies inside the domain of this map. Therefore we get a morphism $f : G/P \to \mathit{IMAGE} \hookrightarrow \mathbb{P}(S^1)$. The space IMAGE is a G-space (a homogeneous space) and we claim the map f is bijective from G/P to IMAGE. Indeed let us inspect the stabilizer Q of the image of $x = P/P$. This is the stabilizer in $(S^1)^*$ of a line L stabilized by P. So Q is a parabolic subgroup containing P and by the classification of parabolic subgroups containing B we only have to check which elements of the Weyl group stabilize L. That is easy, as L has weight $-w_0\lambda$. Note that things would be much more subtle if we needed the scheme theoretic stabilizer [11; I 2.6] of x. We do not need it as we do not claim our bijection is an isomorphism of varieties.

Next, we recall a lemma from algebraic geometry. The lemma is not stated in its full generality but only in a form which will be useful to us. The proof is given in the Appendix (cf. Sublemma A.5.1). We wish to apply it with the line bundle $\mathcal{L} \approx \mathcal{O}(1)$ on IMAGE. Alternatively one may apply Sublemma A.5.1 to the structure sheaf of $\mathrm{Spec}(k[S^1])$ = the affine cone over IMAGE.

Sublemma 4.2.5 *Let X, Y be two projective varieties over k and let $f : X \to Y$ be a morphism which is bijective. Then for every ample line bundle $\mathcal{L}$ on Y and for $s \in H^0(X, f^*(\mathcal{L}))$ we have $s^{p^n} \in H^0(Y, \mathcal{L}^{p^n})$ for some large n.*

Therefore (cf. [7; II 7]) for every $a \in A^1$, we have $a^{p^m} \in S$ for some large m. Now using the p-root closure of S we see that $a \in S$. Thus $A^1 \subset S$, a contradiction. $\square$

Remark 4.2.6 There is another way to understand why $a^{p^m} \in S$ for some large m. Namely, scheme theoretically the stabilizer Q is generated by P and some connected infinitesimal subgroup. This connected infinitesimal subgroup is contained in a Frobenius kernel [11] and thus acts trivially on a^{p^m} for some large m.

4.3 Frobenius Splittings

In this section we define Frobenius splittings and introduce the canonical splittings.

Let R be a k-algebra.

Definition 4.3.1 A *Frobenius-linear endomorphism* of R is a map $\sigma : R \to R$ such that for $a, b \in R$,

(i) $\sigma(a+b) = \sigma(a) + \sigma(b)$

(ii) $\sigma(a^p b) = a \cdot \sigma(b)$

We denote the space of Frobenius-linear endomorphisms by $\mathrm{End}_F(R)$.

Definition 4.3.2 1. A Frobenius-linear endomorphism σ is called a *splitting* if $\sigma(a^p) = a$. This means σ is a splitting if and only if $\sigma(1) = 1$.

2. Let I be an ideal of R. We say a $\sigma \in \mathrm{End}_F(R)$ is *compatible* with I if and only if $\sigma(I) \subset I$. We denote the space of such endomorphisms by $\mathrm{End}_F(R, I)$.

3. We say I is *compatibly split* in R if there exists a splitting σ of R such that $\sigma \in \mathrm{End}_F(R, I)$.

Definition 4.3.3 Let R be a k-algebra. For $a \in R$ and $\sigma \in \mathrm{End}_F(R)$ we define $a * \sigma$ by

$$a * \sigma(b) = \sigma(a \cdot b).$$

Definition 4.3.4 Let $A = \oplus_{i \geq 0} A^i$ be a graded B-algebra. A $\sigma \in \mathrm{End}_F(A)$ is called *graded* if $\sigma(A^{ip}) \subset A^i$ and $\sigma(A^i) = 0$ if i is not divisible by p.

In case R is a G-module, we define a G action on $\mathrm{End}_F(R, I)$ by

$$(g * \sigma)(a) = g \cdot \sigma(g^{-1} \cdot a).$$

Let R be a B-algebra. Then under the $*$ action, the module $\mathrm{End}_F(R)$ is a B-module, possibly not rational. Now B is generated by the torus T and the one-parameter subgroups $U_\alpha = \{x_\alpha(t) \mid t \in k\}$ with α a simple root. Every $\sigma \in \mathrm{End}_F(R)$ defines a map $B \to \mathrm{End}_F(R)$ by $b \mapsto b * \sigma$. If the B-module $\mathrm{End}_F(R)$ is finite dimensional, one expects this to define a polynomial map on each of the subgroups U_α. A T-invariant splitting is canonical if an even stronger condition is true.

Definition 4.3.5 A splitting $\sigma \in \mathrm{End}_F(R)$ (or $\sigma \in \mathrm{End}_F(R, I)$) is called *canonical* if for every simple root α, there exist $\sigma_{r,\alpha} \in \mathrm{End}_F(R)$ such that

(i) $h * \sigma = \sigma$ for every $h \in T(k)$.

(ii) $x_\alpha(t) * \sigma = \displaystyle\sum_{r=0}^{p-1} t^r * \sigma_{r,\alpha}$ for every simple root α and every $t \in k$.

Here it is important that the summation stops at $p-1$.

Remark 4.3.6 If R is a B-algebra and σ a canonical splitting on R, then σ takes weight vectors of weight $p\lambda$ to weight vectors of weight λ. Therefore $\sigma(R_\mu) = 0$ if $\frac{1}{p}\mu$ is not a weight of R.

The following proposition underlines the importance of canonical splittings.

Proposition 4.3.7 (Key Proposition) *Let σ be a canonical splitting of the B-algebra R. Then the image under σ of a B-submodule of R is again B-invariant—and thus a B-submodule.*

Proof: Let v be in a B-submodule N. Recall that one may write $x_\alpha(t)v$ as a polynomial $\sum_{i\geq 0} t^i X_\alpha^{(i)} v$. This explains the notation $X_\alpha^{(i)}$ in what follows.

We write $z(t) = (x_\alpha(-t^p) * \sigma)(v)$ in two ways. On the one hand we have

$$z(t) = \sum_{i\geq 0} (-t^p)^i X_\alpha^{(i)} \sigma \left(\sum_{j\geq 0} t^{jp} X_\alpha^{(j)} v \right) = \sum_{i,j\geq 0} t^{ip+j} (-1)^i X_\alpha^{(i)} \sigma(X_\alpha^{(j)} v).$$

On the other hand, as σ is canonical, $z(t)$ equals

$$\begin{aligned}\sum_{r=0}^{p-1}((-t^p)^r * \sigma_{r,\alpha})(v) &= \sum_{r=0}^{p-1}(\sigma_{r,\alpha}((-t^p)^r v)) \\ &= \sum_{r=0}^{p-1}(-t)^r(\sigma_{\alpha,r}(v)).\end{aligned}$$

Write $z(t) = \sum_{n\geq 0} z_n t^n$. Then $\sigma(v) = z_0$. From the second expression one sees that the other z_{pn} vanish, so $\sigma(v) = \sum_{n\geq 0} z_{pn} t^{pn}$. Now we use the first expression to rewrite this as

$$\sum_{i,s\geq 0} t^{ip+sp}(-1)^i X_\alpha^{(i)} \sigma(X_\alpha^{(ps)} v).$$

But that is just

$$x_\alpha(-t^p)\sum_{s\geq 0}\sigma((t^p)^{ps}X_\alpha^{(ps)}v),$$

whence the result that $x_\alpha(t^p)\sigma(v)$ is in $\sigma(N)$. Now just substitute t for t^p. (We have t vary over an algebraically closed field.) We conclude that $\sigma(N)$ is invariant under all $x_\alpha(t)$ with α simple. It is more or less built into the definition of canonical that $\sigma(N)$ is also invariant under $T(k)$. Now use that $B(k)$ is generated by $T(k)$ and the above $x_\alpha(t)$. □

This proposition together with Remark 4.3.6 immediately gives us the following corollary. Here $A_{<\lambda}$ is the obvious variation on $A_{\leq\lambda}$. It equals $\bigoplus_i A^i_{<i\lambda}$, where $A^i_{<i\lambda}$ is the largest B-submodule of A^i which is in the category $\mathcal{C}_{<i\lambda}$ consisting of all B-modules with weights strictly preceding $i\lambda$ in length–height order.

Corollary 4.3.8 *If σ is a graded canonical splitting on A, then we have $\sigma(A_{\leq\lambda}) \subseteq A_{\leq\lambda}$ and $\sigma(A_{<\lambda}) \subseteq A_{<\lambda}$.* □

The Remark 4.2.1 motivates us to look for geometric examples of splittings and in particular canonical splittings. The Frobenius-linear endomorphisms have the following geometric extension.

Let X be a variety over k. Let $F : X \to X$ denote the *absolute Frobenius morphism*, *i.e.* the morphism which on $\mathcal{O}_X$ restricts to the morphism induced by taking pth power. This morphism is identity on the underlying topological space. However, on functions, it takes a given function to its pth power.

We define $\mathcal{E}nd_F(X)$ — sheaf of Frobenius-linear endomorphisms — by assigning the abelian group $\mathrm{End}_F(\mathcal{O}_X(U))$ to each open U. Let $F_*\mathcal{O}_X$ be the direct image of $\mathcal{O}_X$. As a sheaf of abelian groups, the sheaf $F_*\mathcal{O}_X$ is isomorphic to $\mathcal{O}_X$. The $\mathcal{O}_X$-module structure of $F_*\mathcal{O}_X$ is via the Frobenius morphism. We therefore have $a \cdot s = a^p s$ for $a \in \mathcal{O}_X$ and $s \in F_*\mathcal{O}_X$. Thus, $\mathcal{E}nd_F(X) = (F_*\mathcal{O}_X)^*$, the dual of $F_*\mathcal{O}_X$. This gives an $\mathcal{O}_X$-module structure on $\mathcal{E}nd_F(X)$. We denote the space of global sections of $\mathcal{E}nd_F(X)$ by $\mathrm{End}_F(X)$. We get

$$\begin{aligned}\mathrm{End}_F(X) &= H^0(X, \mathcal{E}nd_F(X)) \\ &= H^0(X, (F_*\mathcal{O}_X)^*).\end{aligned}$$

Definition 4.3.9 A variety X over k is called Frobenius split if there exists $\sigma \in \mathrm{End}_F(X)$ which is a splitting.

If X is a G-variety, we can give a G-structure to $\mathrm{End}_F(X)$ by $(g * \sigma)(s) = g \cdot \sigma(g^{-1} \cdot s)$ for $s \in \mathcal{O}_X$.

The operation $*$ defined before gives another $\mathcal{O}_X$-module structure on $\mathcal{E}nd_F(X)$. We see that this $\mathcal{O}_X$-module structure is obtained by using the isomorphism between $F_*\mathcal{O}_X$ and $\mathcal{O}_X$ as abelian groups. If X is smooth, then the sheaf $\mathcal{E}nd_F(X)$ is isomorphic to a line bundle under the $*$ operation. This is best seen by passing to the completion at a point, which makes things very computable. (Recall the completion of the local ring at a smooth point is just a power series ring in a number of variables.)

Let Y be a closed subvariety of X. Let $\mathcal{I}$ be the sheaf of ideals defining Y. We define the sheaf of Frobenius-linear endomorphisms which are *compatible* with Y by assigning the abelian group $\mathrm{End}_F(\mathcal{O}_X(U), \mathcal{I}(U))$ to any open subset U of X. We denote this sheaf by $\mathcal{E}nd_F(X, Y)$ and its space of global sections by $\mathrm{End}_F(X, Y)$.

Definition 4.3.10 A closed subvariety Y is said to be compatibly split in X if there exists a splitting $\sigma \in \mathrm{End}_F(X, Y)$.

We next list certain properties of splittings and canonical splittings which are useful to us.

Direct images:

1. Let $f : Z \to X$ be a morphism such that $f_* \mathcal{O}_Z = \mathcal{O}_X$. Suppose σ is a splitting on Z such that σ compatibly splits $Y \subset Z$. Then there exists a splitting on X which compatibly splits $f(Y)$. $\square$

2. If moreover Z, Y, X are B-varieties, f is a B-equivariant morphism and $\sigma \in \mathrm{End}_F(Z, Y)$ is canonical, then the induced splitting in $\mathrm{End}_F(X, f(Y))$ is also canonical. $\square$

Lemma 4.3.11 *Let $\sigma \in \mathrm{End}_F(X)$ be a splitting and $\mathcal{L}$ a line bundle on X. Then σ extends uniquely to a graded splitting of $R(\mathcal{L}) = \oplus_{i \geq 0} H^0(X, \mathcal{L}^i)$.*

Proof: Let $V \subset X$ be such that $V = \mathrm{Spec} A$ is affine and $\mathcal{L}|_V$ is trivial. Then $R(\mathcal{L})$ is a polynomial ring $A[T]$. We first prove that a splitting of A extends uniquely to a graded splitting of $A[T]$. We define $\widetilde{\sigma}_V$ by $\widetilde{\sigma}_V(\sum_{i \geq 0} a_i T^i) = \sum_{i \geq 0} \sigma(a_{ip}) T^i$. It is clear that any splitting of $A[T]$ which restricts to σ on A and which is graded has to satisfy this equation. Therefore this extension is unique. It is this uniqueness that allows us to patch these local sections $\widetilde{\sigma}_V$ to get a splitting of $R(\mathcal{L})$. $\square$

Remark 4.3.12 For a B-variety X and equivariant line bundle $\mathcal{L}$, the extension of a canonical splitting will again be canonical.

Let G be our reductive algebraic group over k, with B (and $T \subset B$) a Borel (and torus) subgroup of G. We now consider the special case of $X = G/B$. We will prove that the Demazure desingularisation Z of G/B, introduced in the second section of the first chapter, has a canonical splitting. Therefore using the direct image property of splittings, G/B itself will have a canonical splitting.

Let W be the Weyl group of G. Let $s_1 \cdots s_n$ be a reduced expression for the longest element w_0 in W. For each s_i, we have a minimal parabolic subgroup P_i of G. Then, $Z_n = P_1 \times^B P_2 \times^B \cdots \times^B P_n / B$ is called the Demazure desingularisation of G/B. The multiplication map $m : P_1 \times \cdots \times P_n \to G$ induces a morphism $\varphi : Z_n \to G/B$. The morphism φ is birational. Thus as G/B is a normal variety, we get $\varphi_* \mathcal{O}_{Z_n} = \mathcal{O}_{G/B}$ ([11; II Lemma 14.5]).

Remark 4.3.13 Later we will also have use for Z_n when n is more than the number of positive roots. Then of course $s_1 \cdots s_n$ will not be a reduced expression for w_0. Much of the discussion that follows applies to this more general situation.

We define divisors $\widetilde{D_j} = P_1 \times^B \cdots \times^B P_{j-1} \times^B B \times^B P_{j+1} \cdots P_n/B$ of Z_n. Let $D_n = \bigcup_{j=1}^n \widetilde{D_j}$. The components of D_n intersect transversally at their intersection point $x = B \times^B \cdots \times^B B/B$.

Consider $\mathcal{E}nd_F(Z_n, D_n)$, the sheaf of Frobenius-linear endomorphisms on Z_n which leave the ideal of D_n invariant. Since D_n is a codimension one subvariety of the smooth variety Z_n, the duality theory for the absolute Frobenius map $F : Z_n \to Z_n$ tells us that $\mathcal{E}nd_F(Z_n, D_n) \approx \omega_{Z_n}(D_n)^{1-p}$. (See also A.3.5, A.4.6). Here ω_{Z_n} denotes the canonical line bundle $\Omega^n_{Z_n}$ of Z_n.

Definition 4.3.14 Let $\mathcal{V}$ be a B-equivariant vector bundle on a variety X with B action. (That is, on the corresponding geometric vector bundle there is a B action compatible with the action on X.) Then $\mathcal{V}[\lambda]$ denotes the same vector bundle, but with B action twisted by λ: For $s \in H^0(U, \mathcal{V})$, $b \in B$, we let $b.s$ be $\lambda(b)$ times what it would be without the twist.

Proposition 4.3.15 *The sheaf $\mathcal{E}nd_F(Z_n, D_n)$ is B-equivariantly isomorphic with $\varphi^*\mathcal{L}((1-p)\rho)[(p-1)\rho]$, so that if $\varphi : Z_n \to G/B$ is surjective, its module of global sections* $\mathrm{End}_F(Z_n, D_n)$ *is B-equivariantly isomorphic with $k_{(p-1)\rho} \otimes H^0(G/B, \mathcal{L}((1-p)\rho))$.*

For the proof we refer reader to the Appendix (A.4.6). □

Restricting the above isomorphism to global sections, we get the following corollary.

Corollary 4.3.16 *If the map $\varphi : Z_n \to G/B$ is surjective, then there exists a B-equivariant isomorphism between* $\mathrm{End}_F(Z_n, D_n)$ *and $k_{(p-1)\rho} \otimes H^0(G/B, \mathcal{L}((1-p)\rho))$.* □

Proposition 4.3.17 *Let $\{s_1, \ldots, s_n\}$ denote a sequence of simple reflections, let P_i be the corresponding minimal parabolic subgroups and let $Z_n = P_1 \times^B \cdots \times^B P_n/B$ be as above. Let $\varphi : Z_n \to G/B$ be the "multiplication" map which we assume to be surjective. Then there exists $\sigma \in$* $\mathrm{End}_F(Z_n, D_n)$ *which is a canonical splitting.*

Remark 4.3.18 The surjectivity is not really needed for the conclusion to hold.

Proof of Proposition 4.3.17: (See also the Appendix A.4.7.) To get a candidate for the canonical splitting we use [24] to which we refer for details. As Mehta and Ramanathan explain in [24], one gets a splitting by taking the correct scalar multiple of any element of $\mathrm{End}_F(Z_n, D_n)$ that does not vanish at the intersection point x of the components of D_n. And such an element can be obtained by pulling back a section of $\mathcal{L}((1-p)\rho)[(p-1)\rho]$ that does not vanish at B/B. We claim the splitting may be taken to be T-equivariant so that it satisfies the first condition for being canonical. Indeed, if it were not T-equivariant we could simply take its weight zero component and we would find that that component is also a splitting (exercise). From Proposition 4.3.15 we see that the weight zero space of $\mathrm{End}_F(Z_n, D_n)$ is one-dimensional, so in fact we end up with a unique splitting this way. Now the extremal weights of $H^0(G/B, \mathcal{L}((1-p)\rho))$ are in the Weyl group orbit of $(1-p)\rho$, so for a simple root α the ladder $\{i\alpha \mid i\alpha \text{ is a weight of } \mathrm{End}_F(Z_N, D_N)\}$ stops with $(p-1)\alpha = (p-1)\rho - s_\alpha(p-1)\rho$. Thus the second condition for being canonical is also satisfied. □

4.4 Donkin's Conjecture

In this section we prove Donkin's conjecture.

Theorem 4.4.1 (Canonical splittings and good filtrations) *Let A be a connected (i.e. $A^0 = k$), graded B-algebra with excellent filtration. Let σ be a graded canonical splitting of A. If S is a graded σ-invariant subalgebra which is a G-module, then S has a good filtration.*

Proof: We concentrate on proving that S^1 has a good filtration. The other degrees can be treated similarly, using rescaling as in the proof of 4.2.3. (We ask the reader to figure out how a graded canonical splitting on A defines one on the rescaled algebra $\bigoplus_i A^{m\cdot i}$.)

The length–height filtration of A is an excellent filtration, therefore $A_{\leq\lambda}$ also has excellent filtration. For any weight λ of A, the B-subalgebra $A_{\leq\lambda}$ of A is invariant under the canonical splitting, as is its ideal $A_{<\lambda}$ (Corollary 4.3.8).

Also note for $\lambda \in X(T)^+$, that the submodule $S \bigcap A_{\leq\lambda}$ is again G-invariant, as is its ideal $S \bigcap A_{<\lambda}$ (cf. 4.1.2). We therefore replace A by $A_{\leq\lambda}/A_{<\lambda}$—with its induced canonical splitting—and S by $S \bigcap A_{\leq\lambda}/S \bigcap A_{<\lambda}$. Then with these new choices λ is such that $i\lambda$ is the only weight in $\operatorname{soc} A^i$. Also S is p-root closed since S is invariant under σ and $\sigma(a^p) = a$. We now use Theorem 4.2.3 to see that S has good filtration. □

Next, we give a geometric implication of the above theorem. Note that the motivating variety is $G \times^B G/B$.

Lemma 4.4.2 *Let X be a B-variety and Y a B-invariant subvariety. Let $G \times^B X$ denote the associated fibre bundle over G/B with fibre X. Assume that there exists a canonical splitting σ of $G \times^B X$ compatible with $G \times^B Y$. Let $\mathcal{L}$ be a G-equivariant line bundle on $G \times^B X$. Let $K(\mathcal{L})$ denote the kernel of the restriction morphism* res : $H^0(G \times^B X, \mathcal{L}) \to H^0(G \times^B Y, \mathcal{L})$. *Then the G-modules $H^0(G \times^B X, \mathcal{L})$ and $K(\mathcal{L})$ have good filtrations.*

Proof: Let $\pi : G \times^B X \to G/B$ be the projection map. Now $\oplus_n H^0(G \times^B X, \mathcal{L}^n) \hookrightarrow \oplus_n H^0(\pi^{-1}(Bw_0B/B), \mathcal{L}^n)$. But $\pi^{-1}(Bw_0B/B) \approx Bw_0T \times^T X$ in a B-equivariant way and therefore $H^0(\pi^{-1}(Bw_0B)/B, \mathcal{L}^n) = \operatorname{ind}_T^B H^0(w_0T \times^T X, \mathcal{L}^n)$. Therefore $\oplus H^0(\pi^{-1}(Bw_0B/B), \mathcal{L}^n)$ is an injective B-module. Thus by the cohomological criterion, (Theorem 3.2.7), it has an excellent filtration. Now we extend σ to a graded canonical splitting on $\oplus_n H^0(\pi^{-1}(Bw_0B), \mathcal{L}^n)$. This splitting leaves the G-submodule $\oplus_n H^0(G \times^B X, \mathcal{L}^n)$ invariant. Therefore, by Theorem 4.4.1, $\oplus H^0(G \times^B X, \mathcal{L}^n)$ has good filtration. Therefore $H^0(G \times^B X, \mathcal{L})$ has good filtration.

Similar arguments show that the G-module $H^0(G \times^B Y, \mathcal{L})$ has good filtration.

Consider next the following diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\oplus_n K(\mathcal{L}^n) & \to & \oplus_n K'(\mathcal{L}^n) \\
\downarrow & & \downarrow \\
\oplus_n H^0(G \times^B X, \mathcal{L}^n) & \hookrightarrow & \oplus_n H^0(Bw_0B \times^B X, \mathcal{L}^n) \\
\downarrow \text{res} & & \downarrow \text{res} \\
\oplus_n H^0(G \times^B Y, \mathcal{L}^n) & \hookrightarrow & \oplus_n H^0(Bw_0B \times^B Y, \mathcal{L}^n)
\end{array}$$

Now the splitting on $\oplus_n H^0(Bw_0B \times^B X, \mathcal{L}^n)$ restricts to a splitting on the algebra $k \oplus \bigoplus_n K'(\mathcal{L}^n)$. (We added k in degree zero to get an algebra rather than an ideal.) Further, this splitting leaves $k \oplus \bigoplus_n K(\mathcal{L}^n)$ invariant. Therefore, by Theorem 4.4.1, $K(\mathcal{L})$ also has good filtration. $\square$

Now we are in a position to prove Donkin's conjecture. Like all main results in these notes, it and its method of proof are due to Mathieu. (The reader is invited to compare our exposition with that of Mathieu, to see where the emphasis differs.)

Theorem 4.4.3 (Donkin's Conjecture) *Let $\lambda, \mu \in X(T)^+$.*

1. *$P(\lambda) \otimes P(\mu)$ has a good filtration.*

2. **(Restriction Conjecture)** *Let L be the Levi factor of a parabolic subgroup P of G and let $\lambda \in X(T)^-$. Then $\mathrm{res}^G_L(\mathrm{ind}^G_B(\lambda))$ as an L-module has a good filtration.*

Proof: We are now in a position to exploit Remark 4.2.1 We have $P(\lambda) \otimes P(\mu) = H^0(G \times^B G/B, (G \times^B \mathcal{L}(\mu))[\lambda])$ with $G \times^B G/B \approx G/B \times G/B$. If $s_1, \ldots, s_n$ is a sequence of simple reflections such that—with suitable choice of φ—the map $Z_n = P_1 \times^B \cdots \times^B P_n/B \xrightarrow{\varphi} G \times^B G/B$ is birational, then we have a canonical splitting on Z_n inducing one on $G \times^B G/B$ and by Lemma 4.4.2 we get the first result.

For the second result notice that similarly $P \times^B G/B \approx P/B \times G/B$. The module we have to study is now the restriction of $H^0(P \times^B G/B, (P \times^B \mathcal{L}(\lambda))$. As P/B is a Schubert variety, it has its Demazure resolution just like G/B. It is thus not difficult to come up with $s_1, \ldots, s_n$, such that $Z_n = P_1 \times^B \cdots \times^B P_n/B \xrightarrow{\varphi} P \times^B G/B$ is birational. Thus $P \times^B G/B = L \times^{L \cap B} G/B$ has a canonical splitting, which of course remains canonical with respect to the Borel subgroup $B \cap L$ of L. Apply Lemma 4.4.2. $\square$

Exercise 4.4.4 Read the Appendix and fill in the details in the above proof.

Chapter 5

Joseph's Conjecture

In the last chapter we proved that the tensor product of two modules with good filtrations has good filtration. Now as the reader will see, (Example 5.3.1), the tensor product of two modules with excellent filtration need not have excellent filtration. However in this chapter we prove Joseph's conjecture which says that the tensor product of a module with good filtration and an anti-dominant character has excellent filtration.

We will prove that $\lambda \otimes P(\mu) \otimes Q(\nu)$ is B-acyclic for $\lambda \in X(T)^-$, $\mu \in X(T)^+$ and $\nu \in X(T)$. This implies, by the cohomological criterion, that the tensor product $\lambda \otimes P(\mu)$ has excellent filtration for λ anti-dominant and μ dominant. From this the Joseph's conjecture follows.

To prove the vanishing of B-cohomology, we first induce these modules up to G using the ind_B^G functor. We then prove that the induced G-modules have good filtration and thereby are G-acyclic. We then use the Frobenius reciprocity to prove the B-acyclicity. The use of Frobenius reciprocity requires the ind_B^G-acyclicity of these modules and we use the method of Frobenius splitting to prove the same.

5.1 Double Schubert Varieties

Let $w, z \in W$ be two elements of the Weyl group of G. Let P and Q be two parabolic subgroups of G, containing B. Let X_w and X_z denote the Schubert varieties $\overline{BwP}/P \subset G/P$ and $\overline{BzQ}/Q \subset G/Q$ respectively. Consider the closed B-subvariety $X_w \times X_z$ of $G/P \times G/Q$.

Definition 5.1.1 By a double Schubert variety we mean the subvariety $G \times^B (X_w \times X_z)$ of $G \times^B (G/P \times G/Q)$.

As the total space of the fibre bundle $G \times^B (G/P \times G/Q)$ is isomorphic with $G/B \times G/P \times G/Q$, a double Schubert variety is naturally embedded in the triple product $G/B \times G/P \times G/Q$.

Proposition 5.1.2 *There exists a canonical splitting of $G/B \times G/P \times G/Q$ such that all double Schubert varieties are simultaneously compatibly split in the triple product.*

For the proof we refer the reader to the Appendix (Proposition A.4.8).

Let $\mu \in X(T)^-$, and let P_μ be the parabolic subgroup such that μ extends to a character on P_μ and it is maximal for this property. Therefore on G/P_μ, the line bundle $\mathcal{L}(\mu)$ associated to the character μ exists and is ample. Indeed we work with G/P_μ instead of G/B for precisely this reason. One further notes that if $\pi : G/B \to G/P_\mu$ is the natural projection map, then we have $\pi^*\mathcal{L}(\mu) = \mathcal{L}(\mu)$.

We have $P_\mu = B$ if and only if μ is regular in $X(T)^-$.

Let λ, μ, ν be characters in $X(T)^-$ with λ regular. Let $\mathcal{L}(\lambda, \mu, \nu)$ denote the line bundle $\mathcal{L}(\lambda) \times \mathcal{L}(\mu) \times \mathcal{L}(\nu)$ on the product $G/B \times G/P_\mu \times G/P_\nu$. Let Σ_1 and Σ_2 be unions of double Schubert varieties in $G/B \times G/P_\mu \times G/P_\nu$ such that $\Sigma_2 \subset \Sigma_1$. Then, we have

Lemma 5.1.3 (i) *$H^0(\Sigma_1, \mathcal{L}(\lambda, \mu, \nu))$ has good filtration.*

(ii) *The restriction map $H^0(\Sigma_1, \mathcal{L}(\lambda, \mu, \nu)) \to H^0(\Sigma_2, \mathcal{L}(\lambda, \mu, \nu))$ is surjective. Further, its kernel $K(\Sigma_1, \Sigma_2, \lambda, \mu, \nu)$ has good filtration.*

Proof: (i) By Corollary 4.4.2 we see that $H^0(\Sigma_1, \mathcal{L}(\lambda, \mu, \nu))$ and the kernel of the restriction map $K(\Sigma_1, \Sigma_2, \lambda, \mu, \nu)$ have good filtration.

(ii) As $G \times^B \Sigma_i$ are compatibly split in $G \times^B (G/P_\mu \times G/P_\nu)$ and the line bundle $\mathcal{L}(\lambda, \mu, \nu)$ is ample on $G/B \times G/P_\mu \times G/P_\nu$ the surjectivity of the restriction map follows from the Appendix (see Corollary A.2.2). □

Remark 5.1.4 Consider the short exact sequence

$$0 \to K(\Sigma_1, \Sigma_2, \lambda, \mu, \nu) \to H^0(\Sigma_1, \mathcal{L}(\lambda, \mu, \nu)) \to H^0(\Sigma_2, \mathcal{L}(\lambda, \mu, \nu)) \to 0.$$

As the kernel $K(\Sigma_1, \Sigma_2, \lambda, \mu, \nu)$ has good filtration, it is G-acyclic. Thus $H^1(G, K(\Sigma_1, \Sigma_2, \lambda, \mu, \nu)) = 0$. Therefore, by writing out the long exact sequence of G-cohomologies corresponding to the above short exact sequence, we get the surjectivity of the restriction map on the G-invariants $H^0(G, \Sigma_1, \mathcal{L}(\lambda, \mu, \nu)) \to H^0(G, \Sigma_2, \mathcal{L}(\lambda, \mu, \nu))$.

The double Schubert varieties arise naturally in the context of filtrations of B-modules in the following manner:

Let λ, μ, ν be characters. Let $M = \lambda \otimes P(\mu) \otimes P(\nu)$. As a vector space M is isomorphic with $P(\mu) \otimes P(\nu)$ but the B action on M is shifted by the character λ.

Let $\mu_1 = w_\mu^{-1}\mu$ and $\nu_1 = w_\nu^{-1}\nu$ be the anti-dominant characters in the respective Weyl group orbits. We put $P = P_{\mu_1}$ and $Q = P_{\nu_1}$.

Using the double Schubert varieties we get the following description of $\mathrm{ind}_B^G(M)$.

Let S be the product $X_{w_\mu} \times X_{w_\nu}$ in $G/P_\mu \times G/P_\nu$. Consider the restricted fibration $f = \pi \circ i$ on G/B as given below.

$$\begin{tikzcd} G \times^B S \arrow[r, hook, "i"] \arrow[dr, "f"'] & G \times^B (G/P_\mu \times G/P_\nu) \arrow[d, "\pi"] \\ & G/B \end{tikzcd}$$

If $\mathcal{L}(M)$ denotes the vector bundle on G/B corresponding to the B-representation M, we have $\mathcal{L}(M) = f_* i^* \mathcal{L}(\lambda, \mu_1, \nu_1)$. Therefore we have

$$\begin{aligned} \mathrm{ind}_B^G(M) &= H^0(G/B, f_* i^* \mathcal{L}(\lambda, \mu_1, \nu_1)) \\ &= H^0(G \times^B S, \mathcal{L}(\lambda, \mu_1, \nu_1)) \end{aligned}$$

If we assume λ regular anti-dominant, the line bundle $\mathcal{L}(\lambda, \mu_1, \nu_1)$ is ample on $G/B \times G/P \times G/Q$. Further, $G \times^B S$ is compatibly split in $G/B \times G/P \times G/Q$. Therefore, we have

$$\begin{aligned} R^j\, \mathrm{ind}_B^G(M) &= H^j(G/B, \mathcal{L}(M)) \\ &= H^j(G \times^B S, \mathcal{L}(\lambda, \mu_1, \nu_1)) \qquad \text{by Remark A.2.8,} \\ &= 0 \qquad \text{for } j > 0 \text{ by Corollary A.2.2.} \end{aligned}$$

Thus we have the following lemma.

Lemma 5.1.5 *Let $\lambda \in X(T)^-$ be regular. Let S be a union of products of Schubert varieties in $G/P_\mu \times G/P_\nu$ with $\mu, \nu \in X(T)^-$. Then, $M = \lambda \otimes H^0(S, \mathcal{L}(\mu) \times \mathcal{L}(\nu))$ is ind_B^G-acyclic.*

Proof: The above reasoning also works for such a union. $\square$

5.2 Joseph's Conjecture

In this section we will prove Joseph's conjecture. Moreover, for a regular, anti-dominant character λ and any two characters μ and ν we will prove the B-acyclicity of $\lambda \otimes Q(\mu) \otimes Q(\nu)$.

Lemma 5.1.5 gives us the following vanishing result.

Lemma 5.2.1 *Let λ, μ, ν be anti-dominant with λ being regular. Let S, S_1, S_2 be unions of products of Schubert varieties with $S_2 \subset S_1$. Then*

(i) *$M = \lambda \otimes H^0(S, \mathcal{L}(\mu) \times \mathcal{L}(\nu))$ is B-acyclic.*

(ii) *$M' = \mathrm{Ker}\{\lambda \otimes H^0(S_1, \mathcal{L}(\mu) \times \mathcal{L}(\nu)) \to \lambda \otimes H^0(S_2, \mathcal{L}(\mu) \times \mathcal{L}(\nu))\}$ is B-acyclic.*

Proof: (i) By Lemma 5.1.3 we see that $\mathrm{ind}_B^G(M)$ has good filtration. Further, by Lemma 5.1.5 M is ind_B^G-acyclic. Therefore, we have $H^i(B, M) = H^i(G, \mathrm{ind}_B^G(M)) = 0$.

(ii) We know that both $\lambda \otimes H^0(S_i, \mathcal{L}(\mu) \times \mathcal{L}(\nu))$ are B-acyclic. Further, using Remark 5.1.4 and Frobenius reciprocity, we see that $H^0(B, \lambda \otimes H^0(S_1, \mathcal{L}(\mu) \times \mathcal{L}(\nu))) \to H^0(B, \lambda \otimes H^0(S_2, \mathcal{L}(\mu) \times \mathcal{L}(\nu)))$ is surjective. Now we write the long exact sequence of B-cohomology associated with $0 \to M' \to \lambda \otimes H^0(S_1, \mathcal{L}(\mu) \times \mathcal{L}(\nu)) \to \lambda \otimes H^0(S_2, \mathcal{L}(\mu) \times \mathcal{L}(\nu)) \to 0$ to get the result. $\square$

Corollary 5.2.2 *Let $\lambda \in X(T)^-$ be regular. Let $\mu, \nu \in X(T)$ and let $Q(\mu)$, $Q(\nu)$ denote the relative Schubert modules with socle μ and ν respectively. Then, $\lambda \otimes Q(\mu) \otimes Q(\nu)$ is B-acyclic.*

Proof: Recall that the relative Schubert modules $Q(\mu)$ are defined as kernels of the restriction map of $P(\mu)$ onto the sections over the boundary of the Schubert variety defining $P(\mu)$. We take $S_1 = X_{w_\mu} \times X_{w_\nu}$ and $S_2 = (\partial X_{w_\mu} \times X_{w_\nu}) \cup (X_{w_\mu} \times \partial X_{w_\nu})$. Then the kernel of the restriction map $\lambda \otimes H^0(S_1, \mathcal{L}(\mu_1) \times \mathcal{L}(\nu_1)) \to \lambda \otimes H^0(S_2, \mathcal{L}(\mu_1) \times \mathcal{L}(\nu_1))$ is canonically isomorphic with $\lambda \otimes Q(\mu) \otimes Q(\nu)$ where μ_1 and ν_1 are the anti-dominant characters in the Weyl group orbit of μ and ν respectively. Now the Lemma 5.2.1 gives the result. $\square$

Corollary 5.2.3 *Let $\lambda \in X(T)^-$ be regular and let μ be any character. Then $\lambda \otimes Q(\mu)$ has excellent filtration.*

Proof: Apply the cohomological criterion for excellent filtration (Theorem 3.2.7). $\square$

In order to prove Joseph's conjecture we now need the following lemma.

Lemma 5.2.4 *Let ρ be the character corresponding to the half sum of positive roots. Then, for $\lambda \in X(T)^+$ we have $k_\rho \otimes P(\lambda) = Q(\lambda + \rho)$.*

Proof: We have a natural multiplication map from $H^0(G/B, \mathcal{L}(w_0\lambda) \otimes H^0(G/B, \mathcal{L}(-\rho))$ to $H^0(G/B, \mathcal{L}(w_0\lambda) \otimes \mathcal{L}(-\rho))$. Let k_ρ be the weight space of weight ρ of $H^0(G/B, \mathcal{L}(-\rho))$. We restrict the multiplication map to the subspace $H^0(G/B, \mathcal{L}(w_0\lambda)) \otimes k_\rho$. This gives us a map $m : P(\lambda) \otimes k_\rho \to P(\lambda + \rho)$. This map is injective as it is injective on the one-dimensional socle of its domain. (Use the geometric description of extremal weights.)

We claim that m defines a natural isomorphism between $P(\lambda) \otimes k_\rho$ and $Q(\lambda + \rho) \subset P(\lambda + \rho)$.

To see this we first fix a non-zero element $f \in k_\rho \subset H^0(G/B, \mathcal{L}(-\rho))$. Then f vanishes on lower dimensional Schubert varieties X_w. Thus the image of the multiplication map m is contained in $Q(\lambda + \rho)$.

To see the surjectivity, view $1/f$ as a rational section of $\mathcal{L}(\rho)$. Notice that $1/f$ has pole of order 1 along the codimension one Schubert varieties (5.2.5). Now if $\mathcal{L}$ is a line bundle and s any section of $\mathcal{L}$, we get a (possibly rational) section s/f of the line bundle $\mathcal{L} \otimes \mathcal{L}(\rho)$. Thus, for a section s of the line bundle $\mathcal{L}(w_0\lambda - \rho)$, the element s/f gives us a rational section of $\mathcal{L}(w_0\lambda)$. However, if we restrict this map to the subspace $Q(\lambda+\rho)$ of $P(\lambda+\rho) = H^0(G/B, \mathcal{L}(w_0\lambda - \rho))$ we get an algebraic map as all the elements of $Q(\lambda+\rho)$ vanish on the codimension one Schubert varieties. This map from $Q(\lambda+\rho)$ to $P(\lambda)$ is injective (by its injectivity on the socle). Therefore the dimensions satisfy

$$\dim_k P(\lambda) \otimes k_\rho = \dim_k P(\lambda) \geq \dim_k Q(\lambda+\rho).$$

Therefore the multiplication map defined above is also surjective. $\square$

The reader is advised to do the following illuminating exercise to see the "geometry" involved in the apparently representation theoretic lemma above. The exact formula for computing the degree of a line bundle $\mathcal{L}(\lambda)$ on G/B restricted to any line of the type P_s/B can be found in [3].

Exercise 5.2.5 (cf. [14]) Let $f \in k_\rho \subset H^0(G/B, \mathcal{L}(-\rho))$ be as in the proof of 5.2.4. Let s be a simple reflection with corresponding minimal parabolic P_s. Show

(i) The restriction of $\mathcal{L}(-\rho)$ to the line P_s/B has degree 1, and the same is true for the restriction to any left translate of P_s/B in G/B.

(ii) The line w_0P_s/B intersects the zero set of f only in the point w_0sB/B.

(iii) f vanishes to order one along the codimension one Schubert variety X_{w_0s}.

We now prove Joseph's conjecture. The proof given here differs a little from the one by Mathieu.

Proposition 5.2.6 (Joseph's Conjecture) *Let $\lambda \in X(T)^-$ and $\mu \in X(T)^+$. Then $\lambda \otimes P(\mu)$ has excellent filtration.*

Proof: We know that for $\lambda \in X(T)^-$ which is also regular, $\lambda \otimes Q(\mu)$ has excellent filtration. Now,

$$\begin{aligned} \lambda \otimes P(\mu) &= (\lambda-\rho) \otimes \rho \otimes P(\mu) \\ &= (\lambda-\rho) \otimes Q(\mu+\rho). \end{aligned}$$

Further, $\lambda - \rho \in X(T)^-$ is regular. (In fact $\nu - \rho$ is regular anti-dominant if and only if ν is anti-dominant.) Therefore by Corollary 5.2.3 we get the result. $\square$

Corollary 5.2.7 (Joseph) *Let $\lambda \in X(T)$ and $\mu \in X(T)^+$. Then, $P(\lambda) \otimes P(\mu)$ has excellent filtration.*

Proof: Let $w \in W$ be such that $w^{-1}\lambda = \nu \in X(T)^-$. We have $P(\nu) = k_\nu$. Therefore, $P(\nu) \otimes P(\mu)$ has excellent filtration. Since μ is dominant, $P(\mu)$ is a G-module and therefore, by the tensor identity, $\mathrm{ind}_B^{P_s}(P(\tau) \otimes P(\mu)) = \left(\mathrm{ind}_B^{P_s} P(\tau)\right) \otimes P(\mu)$ for any simple reflection s and weight τ. Recall that we have $H_s \circ H_z = H_{sz}$ for Joseph functors when the length of sz is more than the length of z. Therefore we see that $H_w(P(\nu) \otimes P(\mu)) = P(w\nu) \otimes P(\nu)$. Now recall that Proposition 3.2.11 states that H_w sends a module with excellent filtration to a module with excellent filtration. Therefore the result. $\square$

For an application of Joseph's conjecture see [21; Theorem 5.5], which gives the existence of a "good basis" in a module with good filtration. One easily checks that although the proof refers to Polo's conjecture (cf. next chapter), it suffices to apply Joseph's conjecture.

5.3 An Example

In this section we give an example showing that the tensor product of modules with excellent filtration need not have have excellent filtration.

Example 5.3.1 We take $G = \mathrm{SL}(3, k)$, with B the subgroup of upper triangular matrices, T the subgroup of diagonal matrices. Inside the G-module M_3 of 3-by-3 matrices, upon which G acts by conjugation, we consider the five-dimensional B-submodule E generated, as a B-module, by the matrices

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It has a four-dimensional submodule S generated by D, and the extension

$$0 \longrightarrow S \longrightarrow E \longrightarrow k \longrightarrow 0$$

does not split. So $H^1(B, S) \neq 0$. Now one checks that $S = P(-s_2\omega_1) \otimes P(-s_1\omega_2) \otimes Q(\rho)$, where ω_1, ω_2 denote the fundamental weights, cf. [11].

(Recall $Q(\rho) = k_\rho$.) So S gives an example of a tensor product of the form $P(\lambda) \otimes P(\mu) \otimes Q(\nu)$ which is not B-acyclic. From the cohomological criteria it then follows that $P(\lambda) \otimes P(\mu)$ does not have excellent filtration and that $P(\mu) \otimes Q(\nu)$ does not have relative Schubert filtration.

Exercise 5.3.2 (Polo) Compute the characters of the $P(\xi)$ for each weight ξ of $P(-s_2\omega_1) \otimes P(-s_1\omega_2)$ and show that $P(-s_2\omega_1) \otimes P(-s_1\omega_2)$ does not even have the character of any module with excellent filtration. Similarly show that $P(-s_1\omega_2) \otimes Q(\rho)$ does not even have the character of any module with relative Schubert filtration.

Chapter 6

Polo's Conjecture

Let ζ be a character. We denote by ζ_1 (by ζ_0) the anti-dominant (the dominant) character in the Weyl group orbit of ζ. The Joseph Conjecture states that for $\lambda \in X(T)^-$ and $\mu \in X(T)$, the module $\lambda \otimes P(\mu_0)$ has excellent filtration. Here we study a generalization of that conjecture, first stated by P. Polo. It says that for $\lambda \in X(T)^-$ and μ arbitrary, the module $\lambda \otimes P(\mu)$ has excellent filtration. Equivalently, we need to prove that $\lambda \otimes P(\mu) \otimes Q(\nu)$ is B-acyclic.

6.1 Reformulating the Problem Repeatedly

We first look at the case when λ is regular anti-dominant. Consider the following exact sequence:

$$0 \longrightarrow \lambda \otimes K \longrightarrow \lambda \otimes P(\mu_0) \longrightarrow \lambda \otimes P(\mu) \longrightarrow 0 \tag{6.1.1}$$

By Joseph's conjecture $\lambda \otimes P(\mu_0)$ has excellent filtration. The module K has a filtration by relative Schubert modules and for λ regular anti-dominant we already know that $\lambda \otimes Q(\nu)$ has excellent filtration for any character ν. Therefore $\lambda \otimes K$ has excellent filtration. Now, using the long exact sequence of B-cohomology associated to 6.1.1, we see that the module $\lambda \otimes P(\mu)$ also satisfies the cohomological criterion for excellent filtration.

However this method fails when λ is not regular as it is no longer true that $\lambda \otimes Q(\nu)$ has excellent filtration. Indeed, when λ is a trivial character, we see that $Q(\nu)$ cannot have an excellent filtration unless ν is anti-dominant.

To tackle the general case we again resort to the same trick. We first induce the B-modules to G-modules and use the canonical splitting to prove results. But first, we need the following lemma.

Lemma 6.1.2 *Let $\lambda, \mu, \nu \in X(T)^-$ and $w \in W$. Let S be a union of Schubert varieties in G/P_ν. Assume that we can prove (for all such λ, μ, ν, w, S) that the natural restriction map*

$$H^0(B, \lambda \otimes H^0(X_w, \mathcal{L}(\mu)) \otimes P(\nu_0)) \to H^0(B, \lambda \otimes H^0(X_w, \mathcal{L}(\mu)) \otimes H^0(S, \mathcal{L}(\nu))) \tag{6.1.3}$$

is surjective. Then Polo's conjecture is true.

Proof: Let $K = \ker(H^0(G/P_\nu, \mathcal{L}(\nu)) \stackrel{\text{res}}{\to} H^0(S, \mathcal{L}(\nu)))$. Let $M = \lambda \otimes K$. We know by Joseph's conjecture that $\lambda \otimes P(\nu_0)$ has excellent filtration. Therefore, $H^1(B, \lambda \otimes P(\nu_0) \otimes Q(\tau)) = 0$ for all τ. However, $H^0(X_w, \mathcal{L}(\mu))$ has a filtration by relative Schubert modules $Q(\tau)$. Hence, $H^1(B, \lambda \otimes P(\nu_0) \otimes H^0(X_w, \mathcal{L}(\mu))) = 0$. Therefore, for $\mu \in X(T)^-$ and $w \in W$, the surjectivity in 6.1.3 gives $H^1(B, M \otimes H^0(X_w, \mathcal{L}(\mu))) = 0$.

Thus $H^1(B, M \otimes \text{module with excellent filtration}) = 0$. Therefore, M has filtration by relative Schubert modules by the cohomological criterion for relative Schubert filtration (cf. Exercise 3.3.3).

This in turn means that $M \otimes P(\tau)$ is B-acyclic for any $\tau \in X(T)$.

For any $z\nu \in X(T)$, we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \to & K_1 & \longrightarrow & H^0(G/P_\nu, \mathcal{L}(\nu)) & \longrightarrow & H^0(X_z, \mathcal{L}(\nu)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \text{res} & & \\
0 & \to & K_2 & \longrightarrow & H^0(G/P_\nu, \mathcal{L}(\nu)) & \longrightarrow & H^0(\partial X_z, \mathcal{L}(\nu)) & \to & 0
\end{array}$$

Further, K_1 and K_2 satisfy the following exact sequence (cf. Exercise A.2.9):

$$0 \longrightarrow K_1 \longrightarrow K_2 \longrightarrow Q(z\nu) \longrightarrow 0.$$

Now we may take $M = \lambda \otimes K_i$ in the above, so $\lambda \otimes K_i \otimes P(\tau)$ is B-acyclic for any $\tau \in X(T)$ ($i = 1$, 2). But then the quotient $\lambda \otimes Q(z\nu) \otimes P(\tau)$ is B acyclic too, for all λ, $\nu \in X(T)^-$, $z \in W$, $\tau \in X(T)$. This proves the lemma. $\square$

Remark 6.1.4 Note that there is also a slightly different argument to prove the B-acyclicity of $M \otimes P(\mu)$ in the above: If one has that $H^1(B, M \otimes$ module with excellent filtration$) = 0$, then let $I_0 \to I_1 \cdots$ be an injective resolution of $P(\mu)$. Consider the exact sequences

$$0 \longrightarrow \ker(I_n \longrightarrow I_{n+1}) \longrightarrow I_n \longrightarrow \mathrm{im}(I_n) \longrightarrow 0$$

of modules with excellent filtration. Tensoring with M and taking B-invariants gives many short exact sequences and thus $H^i(B, M \otimes P(\mu))$ in fact vanishes for $i > 0$. The advantage of this argument is that it does not need the cohomological criterion for relative Schubert filtrations.

To prove surjectivity of 6.1.3, we first induce both modules up to G and there prove that the map on G-invariants is surjective. The Frobenius reciprocity then gives us the surjectivity on B-invariants.

Recall that $\mathrm{ind}_B^G \left(\lambda \otimes H^0(X_w, \mathcal{L}(\mu)) \otimes H^0(S, \mathcal{L}(\nu))\right) = H^0(G \times^B (X_w \times S), \mathcal{L}(\lambda, \mu, \nu))$. Now the line bundle $\mathcal{L}(\lambda, \mu, \nu)$ is not ample on $G \times^B (G/P_\mu \times G/P_\nu)$, unless λ is regular. Therefore, for all we know now, the restriction map $H^0(G \times^B (X_w \times G/P_\nu), \mathcal{L}(\lambda, \mu, \nu)) \to H^0(G \times^B (X_w \times S), \mathcal{L}(\lambda, \mu, \nu))$ need not be surjective, even though $G \times^B (X_w \times S)$ is compatibly split in the product $G/B \times G/P_\mu \times G/P_\nu$. However, the line bundle $\mathcal{L}(\lambda, \mu, \nu)$ is ample on $G/P_\lambda \times G/P_\mu \times G/P_\nu$. Therefore, we consider the following diagram:

$$\begin{array}{ccc} Z = G \times^B (X_w \times S) & \hookrightarrow & G \times^B (G/P_\mu \times G/P_\nu) \\ & & \downarrow \pi \\ & & G/P_\lambda \times G/P_\mu \times G/P_\nu \end{array}$$

The map π is defined by $(g, x, y) \mapsto (\overline{g}, \overline{gx}, \overline{gy})$, where the "bar" denotes the image of an element of G in the corresponding quotient.

Lemma 6.1.5 *If* $\pi_* \mathcal{O}_Z = \mathcal{O}_{\pi(Z)}$, *then*

$$\begin{array}{c} H^0\left(G, \mathrm{ind}_B^G \left(\lambda \otimes H^0(X_w, \mathcal{L}(\mu)) \otimes P(\nu_0)\right)\right) \\ \downarrow \mathrm{res} \\ H^0\left(G, \mathrm{ind}_B^G \left(\lambda \otimes H^0(X_w, \mathcal{L}(\mu)) \otimes H^0(S, \mathcal{L}(\nu))\right)\right) \end{array}$$

is surjective.

Proof: We have $\mathrm{ind}_B^G\left(\lambda \otimes H^0(X_w, \mathcal{L}(\mu)) \otimes P(\nu_0)\right) = H^0(G \times^B (X_w \times G/P_\nu), \mathcal{L}(\lambda, \mu, \nu))$ and $\mathrm{ind}_B^G\left(\lambda \otimes H^0(X_w, \mathcal{L}(\mu)) \otimes H^0(S, \mathcal{L}(\nu))\right) = H^0(G \times^B (X_w \times S), \mathcal{L}(\lambda, \mu, \nu))$. Consider the map of pairs

$$(G \times^B (G/P_\mu \times G/P_\nu), Z) \xrightarrow{\pi} (G/P_\lambda \times G/P_\mu \times G/P_\nu, \pi(Z))$$

If $\pi|_Z$ has the direct image property $\pi_* \mathcal{O}_Z = \mathcal{O}_{\pi(Z)}$, we have

1. $\pi_*(\mathcal{L}(\lambda, \mu, \nu)|_Z) = \mathcal{L}(\lambda, \mu, \nu)|_{\pi(Z)}$ and therefore, $H^0(Z, \mathcal{L}(\lambda, \mu, \nu)) = H^0(\pi(Z), \mathcal{L}(\lambda, \mu, \nu))$.

2. Further, the canonical splitting on the domain will give us a canonical splitting on $G/P_\lambda \times G/P_\mu \times G/P_\nu$, which compatibly splits $\pi(Z)$.

Now, $\mathcal{L}(\lambda, \mu, \nu) = \mathcal{L}(\lambda) \times \mathcal{L}(\mu) \times \mathcal{L}(\nu)$ is ample on $G/P_\lambda \times G/P_\mu \times G/P_\nu$. Therefore the restriction map $H^0(G/P_\lambda \times G/P_\mu \times G/P_\nu, \mathcal{L}(\lambda, \mu, \nu)) \to H^0(\pi(Z), \mathcal{L}(\lambda, \mu, \nu))$ will be surjective. Further, its kernel will have good filtration. This allows us to apply the Remark 5.1.4 to see that the restriction map on G-invariants is surjective. From this the claim follows as this surjective map factors through $H^0\left(G, \mathrm{ind}_B^G\left(\lambda \otimes H^0(X_w, \mathcal{L}(\mu)) \otimes P(\nu_0)\right)\right)$. $\square$

Therefore to prove Polo's conjecture, we only have to prove that $\pi|_Z$ has the indicated direct image property. Now we remark again that the map π is defined on $X = G/B \times G/P_\mu \times G/P_\nu$ and we have $\pi_* \mathcal{O}_X = \mathcal{O}_{G/P_\lambda \times G/P_\mu \times G/P_\nu}$. Therefore we can "push forward" the canonical splitting of X on to its image. This "pushed splitting" will split the image $\pi(Z)$ of Z.

Consider now the following proposition. The proof of this proposition will be given in the Appendix (A.5.2). We have to explain first what *separable* means for $f : X \to Y$. The relevant notion of separability is somewhat fancy, as our varieties are not irreducible. What it means is that there is a dense subset of y in Y for which there is an $x \in f^{-1}(y)$ so that the tangent map at x is surjective. It is thus some kind of generic smoothness.

Proposition 6.1.6 *Let $f : X \to Y$ be a surjective, separable, proper morphism between two varieties, with connected fibres. We assume that Y is Frobenius split. Then $f_* \mathcal{O}_X = \mathcal{O}_Y$.* $\square$

Let us grant separability for the time being. Thus in order to prove Polo's conjecture it only remains to prove that the fibres of the map $\pi : Z \to G/P_\lambda \times G/P_\mu \times G/P_\nu$ are connected. This topological problem will also be reformulated repeatedly.

The reader is asked to be patient about this roundabout proof. The fact is that, as he/she will come to know in Remark 6.1.10, the statement we want to prove is very similar to some false statements. We have to sneak around all these false statements.

First we note a result, which tells us that having connected fibres and having the direct image property are really the same problem, so that we may switch back and forth between the two at our convenience. Indeed we will later turn around and go back all the way to a problem similar to surjectivity of 6.1.3.

Lemma 6.1.7 [7; *Corollary* 11.3] *Let $f : X \to Y$ be a proper morphism between two varieties and assume $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then all fibres of f are connected.* □

Next note that $G \times^B (G/P_\mu \times G/P_\nu) \overset{\phi}{\approx} G \times^{P_\mu} G \times^B G/P_\nu$. The map ϕ is defined on the product by $\phi(g, \overline{x}, \overline{y}) = (gx, x^{-1}, \overline{y})$.

The image of Z under ϕ is $G \times^{P_\mu} (\overline{P_\mu w^{-1} B} \times^B S)$.

We define $\widetilde{\pi} : G \times^{P_\mu} (\overline{P_\mu w^{-1} B} \times^B S) \to G/P_\mu \times G/P_\lambda \times G/P_\nu$ by $\widetilde{\pi}((g, x, \overline{y})) = (\overline{g}, \overline{gx}, \overline{gxy})$. Up to the isomorphism ϕ, this $\widetilde{\pi}$ is just π.

So our aim is to prove that fibres of $\widetilde{\pi}$ are connected. Using the G-equivariance of $\widetilde{\pi}$ we see that we may restrict $\widetilde{\pi}$ to the subspace $Z' = (P_\mu) \times^{P_\mu} (\overline{P_\mu w^{-1} B} \times^B S) = (e) \times (\overline{P_\mu w^{-1} B} \times^B S)$.

All we need is that the fibres of that restricted map are connected.

The image of Z' is contained in $G/P_\lambda \times G/P_\nu$. Note that as $\overline{P_\mu w^{-1} B}$ is an irreducible two-sided B-invariant closed subvariety of G, we have by Bruhat decomposition some $y \in W$ such that $\overline{P_\mu w^{-1} B} = \overline{ByB}$.

Summing up, we have to show that the map $\overline{ByB} \times^B S \to G/P_\lambda \times G/P_\nu$ has connected fibres.

A fibre of the map $\overline{ByB} \times^B S \to G/P_\lambda \times G/P_\nu$ is simply an intersection of a fibre of $\overline{ByB} \times^B S \to G/P_\lambda$ with a fibre of $\overline{ByB} \times^B S \to G/P_\nu$. We first concentrate on the projection towards G/P_λ.

Proposition 6.1.8 *Let P be a parabolic, $X_w \subset G/B$ a Schubert variety. The non-empty fibres of the projection $X_w \to G/P$ are left translates of Schubert varieties.*

Proof: Using the B-equivariance we may restrict attention to the the fibre of zP/P, where z is a minimal representative in the Weyl group W of the coset $zW(P)$, if $W(P)$ denotes the Weyl group of P. Recall from [10; Proposition 1.10], cf. [1; Ch. IV, §1 Exercice 3] that $l(zu) = l(z) + l(u)$ if $u \in W(P)$, so that $BzBuB = BzuB$. The fibre is thus a union of sets $zBuB/B$, where $u \in W(P)$ is such that $zu \leq w$. Recall also (same source) that w decomposes uniquely as $z'u'$ where z' is a minimal representative of the coset $z'W(P)$ and $u' \in W(P)$. Then a lemma of Deodhar (read $w \in W_Q$ where it says $w \in W/W_Q$, in [16; Lemma 4.4]) says there is a unique maximal u. Then the fibre is $z\overline{BuB}/B$ for that maximal u. □

So what the proposition tells us is that we should prove that the fibres of $g\overline{BuB}\times^B S \to G/P_\nu$ are connected for $g \in G$, $u \in W$. And by G-equivariance we may forget g.

Thus we have to prove

Proposition 6.1.9 *The fibres of the multiplication map $m : \overline{BuB} \times^B S \to G/P_\nu$ are connected.*

Remark 6.1.10 We can now point out a subtlety, which shows that one cannot get by just with generalities about Frobenius splittings. Namely, the proposition fails if $\overline{BuB}$ is replaced by a union of $\overline{BvB}$'s ($v \in W$). This is related to the fact that a tensor product of two modules with excellent filtration need not have an excellent filtration (see Example 5.3.1.)

6.2 The Proof of Polo's Conjecture

Clearly Proposition 6.1.9 presents a smaller problem than the one suggested by Lemma 6.1.5. In this section we prove the Proposition 6.1.9 and thus also:

Theorem 6.2.1 (Mathieu; Polo's Conjecture) *Let $\lambda \in X(T)^-$ and let $\mu \in X(T)$. Then $\lambda \otimes P(\mu)$ has excellent filtration.*

Apart from Proposition 6.1.9 one must also must worry about separability. But fortunately this does not require a thorough understanding of fibres. One only needs to show that the source of our map is a finite union of pieces on which the map to "image piece" is separable. The pieces to take are the $\overline{BuB} \times^B$ (component of S) of Proposition 6.1.9, basically. One easily finds subvarieties that actually map birationally to the image of the piece. We leave it at this sketch for now and return to the proof of Proposition 6.1.9.

We first note that if $u = s_1 \cdots s_n$ is a reduced expression of $u \in W$, then the multiplication map $m : \overline{BuB} \times^B S \to G/P_\nu$ can be lifted to the projection

$$P_{s_1} \times^B \cdots P_{s_n} \times^B S \longrightarrow G/P_\nu.$$

The fibres of this projection map surjectively onto the fibres of m. Further, the study may be broken up into little pieces like this:

$$P_{s_1} \times^B \cdots P_{s_n} \times^B S \longrightarrow P_{s_1} \times^B \cdots P_{s_{n-1}} \times^B P_{s_n} S \longrightarrow G/P_\nu.$$

So the trick is to show (cf. Lemma 6.1.7) that $\psi : P_s \times^B S \longrightarrow P_s S$ does have the direct image property.

Say C is the cokernel of the map $\mathcal{O}_{P_s S} \to \psi_* \mathcal{O}_{P_s \times^B S}$. We need to show that $H^0(P_s S, C \otimes \mathcal{L}(n\nu))$ vanishes for large n. (That will show $C = 0$ by ampleness, cf. [11; II 14.6 (4)].)

Consider the following diagram

$$\begin{array}{ccc} P_s \times^B S & \xrightarrow{\psi} & P_s S \subset G/P_\nu \\ \downarrow{\scriptstyle \pi} & & \\ P_s/B & & \end{array}$$

We have

$$\begin{aligned} H^0(P_s \times^B S, \psi^* \mathcal{L}(n\nu)) &= H^0(P_s/B, \pi_* \psi^* \mathcal{L}(n\nu)) \\ &= H_s(H^0(S, \mathcal{L}(n\nu))). \end{aligned}$$

Therefore, we have a natural injective map $H^0(P_s S, \mathcal{L}(n\nu)) \to H^0(P_s \times^B S, \psi^* \mathcal{L}(n\nu)) = H_s(H^0(S, \mathcal{L}(n\nu)))$. By Exercise A.2.9 the proof of Proposition 6.1.9 will be finished once we have the following lemma.

Lemma 6.2.2 *For any B-invariant closed subset S of G/B, any simple reflection s and $\lambda \in X(T)^-$, the natural map $H^0(P_sS, \mathcal{L}_\lambda) \to H_s(H^0(S, \mathcal{L}_\lambda))$ is an isomorphism.*

Proof: We will prove the lemma by induction on "size" of S. Note that if S is irreducible, *i.e.* when S is a Schubert variety X_w, the image P_sS is either X_{sw} (when $sw > w$) or X_w. In either case the lemma is true. Therefore we assume that the lemma is true if we substitute for S any of its proper B-invariant closed subvarieties.

Now we write S as $X_w \cup S'$, and we may replace S' by $S' \cup \partial X_w$ to make sure we understand $S' \cap X_w$ well. Indeed $S' \cap X_w$ is now ∂X_w (even scheme theoretically by Ramanathan). And of course we mean that X_w, S' are really smaller than S. By the Mayer–Vietoris Lemma 2.2.11 we have an exact sequence $0 \to H^0(S, \mathcal{L}) \to H^0(X_w, \mathcal{L}) \oplus H^0(S', \mathcal{L}) \to H^0(\partial X_w, \mathcal{L}) \to 0$. This gives an exact sequence $0 \to H_s(H^0(S, \mathcal{L})) \to H_s(H^0(X_w, \mathcal{L})) \oplus H_s(H^0(S', \mathcal{L})) \to H_s(H^0(\partial X_w))$.

Thus what remains to be checked is that $P_sS' \cap P_sX_w = P_s\partial X_w$, to make the computation go. If $sw < w$, then $P_s\partial X_w = X_w = P_sX_w$, and $X_w \subset P_sS'$.

If $sw > w$, then $P_sX_w = X_{sw}$ and we need that for $z \in W$, $z \neq w$, $z \neq sw$, $sz \leq sw$ implies $z < w$. (The z to be taken are such that $BzB \subset S'$.) That is indeed so, and a reference is [10; 5.9]. (The reader can take this as an exercise!) □

We still have to explain how to handle the details of the separability issue. We do this in a series of exercises. The reader is assumed to be familiar with standard coordinates in Bruhat cells, as explained for instance in [34; Chapter 10].

Exercise 6.2.3 Let $g : Z \to X$, $f : X \to Y$ be maps between varieties, with g surjective, so that fg is separable. Then f is separable.

Exercise 6.2.4 More generally, let $g_i : Z_i \to X$, $i = 1, \ldots n$, $f : X \to Y$ be maps between varieties, with $\bigcup_i g_i(Z_i) = X$, so that each fg_i is separable to its image. Then f is separable to its image.

Exercise 6.2.5 Let $f : X \to Y$ be a separable P_μ-equivariant map. Then it induces a separable map $G \times^{P_\mu} X \to G \times^{P_\mu} Y$.
(Hint: Use that the fibrations $G \times^{P_\mu} X \to G/P_\mu$ and $G \times^{P_\mu} Y \to G/P_\mu$ are locally trivial.)

Exercise 6.2.6 Let z, u be as in the proof of Proposition 6.1.8, with $P = P_\lambda$ and let C be a component of S. Let U_z be the subgroup of U generated by the root groups U_α with $U_\alpha z \cap P = (e)$. Then $a \mapsto \overline{az}$ maps U_z isomorphically to its image in G/P. Furthermore the rule $(a, b, c) \mapsto (a, \overline{azbc})$ maps $U_z \times \overline{BuB} \times C$ separably to its image in $U_z \times G/P_\nu$.

Hint: Replace $\overline{BuB}$ and C by suitable subvarieties to make to make the map $(b, c) \mapsto \overline{bc}$ birational towards $\overline{BuB}C$ and use the automorphism $(a, \overline{b}) \mapsto (a, \overline{ab})$ of $U_z \times G/P_\nu$.

Exercise 6.2.7 Now check that the map needed in the proof of Polo's conjecture is indeed separable.

6.3 Variations and Questions

We start with an analogue of Donkin's restriction conjecture. Let P be a parabolic subgroup corresponding with a subset I of the simple roots, so that P is generated by B and the $U_{-\alpha}$ with $\alpha \in I$. Let L be the Levi factor of P with Borel subgroup $B \cap L$ generated by T and the U_α with $\alpha \in I$.

Theorem 6.3.1 *If M is a B-module with excellent filtration, then $\mathrm{res}^B_{B\cap L} M$ is a $B \cap L$-module with excellent filtration.*

Remark 6.3.2 Note that one may just as well restrict to $B \cap L'$, where L' is the commutator subgroup of L: Any $B \cap L$-module breaks up into a direct sum of weight spaces for the action of the center of L. These weight spaces are $B \cap L'$-modules and they have excellent filtration as $B \cap L'$-modules if and only if they have one as $B \cap L$-modules. If you wish this is so by definition.

Proof of theorem: We may assume M is finite dimensional. Choose an anti-dominant weight δ whose stabilizer in W is the Weyl group $W(L)$ of L. Thus δ lies in the reflecting hyperplanes of the simple reflections corresponding with the elements of I, but not in the other reflecting hyperplanes (see [9; 1.12]). Let C be the closure of the anti-dominant chamber. Then δ lies in the interior of $\bigcup_{w\in W(L)} wC$. As this union is a cone, it follows that for n sufficiently large $\mu + n\delta$ is in the cone for every weight μ of M. We proceed with such n and study $M \otimes k_{n\delta}$, which has excellent filtration by Polo's conjecture. Now for a $B \cap L$-module having an excellent filtration it does not matter whether one twists by δ: all that changes is the action of the center of L. So we may further assume that all weights of M lie in $\bigcup_{w\in W(L)} wC$. In other words, in the excellent filtration of M all the $P(\lambda)$ that occur have their λ in the $W(L)$-orbit of an element λ_1 of $X(T)^-$. Write $P(\lambda) = H_{s_1}H_{s_2}\cdots H_{s_r}(\lambda_1)$ with the s_i simple reflections that are in $W(L)$. Noting that $P_s/B = P_s \cap L/B \cap L$, we get $\operatorname{res}^B_{B\cap L} P(\lambda) = H^L_{s_1}H^L_{s_2}\cdots H^L_{s_r}(\lambda_1)$, where $H^L_{s_i}$ is the analogue of H_{s_i} in the context of L: $H^L_{s_i} = \operatorname{ind}^{P_s\cap L}_{B\cap L}$. So the restriction property holds for all relevant $P(\lambda)$. $\square$

Exercise 6.3.3 State and prove a similar result for relative Schubert filtrations.

Polo has introduced another notion, viz. that of having a Schubert filtration. We first give the definition, then relate it to other concepts to show that the analogue of Polo's conjecture holds for Schubert filtrations too. (This was proved by Polo under some restrictions.)

Definition 6.3.4 A finite dimensional B-module M has a *Schubert filtration* if and only if there exists a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_r = M$ by B-modules such that $F_i/F_{i-1} = H^0(S_i, \mathcal{L}(\lambda_i))$ for some $\lambda_i \in X(T)^-$. Here the S_i are unions of Schubert varieties and $r \geq 0$.

In [27] Polo proves the following cohomological criterion for having a Schubert filtration. If $\lambda \in X(T)^-$, $y \leq w$ in W, put $K(w, y, \lambda) = \ker P(w\lambda) \to P(y\lambda)$.

Theorem 6.3.5 (Polo) *Let M be a finite dimensional B-module. Then M has a Schubert filtration if and only for all $\lambda \in X(T)^-$ and $y \leq w$ in W the module $M \otimes K(w, y, \lambda)$ is B-acyclic.* $\square$

From this it follows that if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact, and M', M have Schubert filtration, then so does M''.

Clearly, a module with Schubert filtration also has a filtration by relative Schubert modules. Also, if s is a simple reflection and M is a module with Schubert filtration, then M is acyclic for H_s and $H_s(M)$ has Schubert filtration. This follows by imitating the proof of Lemma 3.2.11 with the help of Lemma 6.2.2. From Lemma 6.2.2 one then concludes that in fact a relative Schubert module M is already acyclic for H_s. (Another way to see this is through the formula $H^i_s(M) = H^i(B, H_s(k[B]) \otimes M)$, see [11; I 4.10]. As $k[B]$ is injective, $H_s(k[B])$ has excellent filtration and $H_s(k[B]) \otimes M$ is B-acyclic.) This will be used in the proof of

Proposition 6.3.6 *For a B-module M the following are equivalent.*

(i) *M has a Schubert filtration.*

(ii) *The evaluation map $\mathrm{ind}_B^G(M) \to M$ is surjective, its kernel has a relative Schubert filtration and $\mathrm{ind}_B^G(M)$ has a good filtration.*

(iii) *There is a module with good filtration N and a surjective B-module map $N \to M$ whose kernel has relative Schubert filtration.*

Proof: (Sketchy)

(i) $\Rightarrow$ (ii). If the Schubert filtration of M has just one layer, (ii) follows easily. The general case then follows using acyclicity for induction.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Let K be the kernel of $N \to M$. We must show that $M \otimes K(w, y, \lambda)$ is B-acyclic. As M has relative Schubert filtration, the problem is to show that $H^0(B, M \otimes P(w\lambda)) \to H^0(B, M \otimes P(y\lambda))$ is surjective. It suffices to show that $H^0(B, N \otimes P(w\lambda)) \to H^0(B, M \otimes P(y\lambda))$ is surjective. Now $H^0(B, N \otimes P(w\lambda)) = H^0(G, \mathrm{ind}_B^G(N \otimes P(w\lambda))) = H^0(G, \mathrm{ind}_B^G(N \otimes P(y\lambda))) = H^0(B, N \otimes P(y\lambda))$. But $K \otimes P(y\lambda))$ is B-acyclic. □

Corollary 6.3.7 *Let λ be a dominant or an anti-dominant weight and let M have Schubert filtration. Then $P(\lambda) \otimes M$ has Schubert filtration.*

Proof: Write M as a quotient of a module with good filtration by one with relative Schubert filtration and use that the analogue of the corollary holds for those concepts. □

Remark 6.3.8 Mathieu's proof of this corollary (for anti-dominant λ) was similar to his proof of Polo's conjecture. It did not rely on Polo's conjecture, like ours does.

We now list some open questions which are related to those answered in these notes.

QUESTION 1 We know that excellent tensor excellent need not be excellent, (see Example 5.3.1). No counterexamples are known to the following question: Is excellent tensor excellent relative Schubert? That is, is the tensor product of three modules with excellent filtration B-acyclic?

QUESTION 2 Define that M *preserves excellence* if $M \otimes$ excellent $=$ excellent. Using the cohomological criteria one sees this is equivalent to "preserving the existence of a relative Schubert filtration". (It is equivalent to $M \otimes P(\lambda) \otimes Q(\mu)$ being B-acyclic for all λ, μ.) In particular, it implies that $M \otimes Q(\rho) = M \otimes k_\rho$ has relative Schubert filtration. Mathieu conjectures the converse: M preserves excellence if $M \otimes k_\rho$ has relative Schubert filtration.

Remark 6.3.9 There are many related questions one may ask. We do not know for what tensor products one should expect B-acyclicity. It undoubtedly has to do with the facets the weights of the socles lie on.

Chapter 7

Other Base Rings

In this chapter we state the earlier results in their proper generality: The base ring need not be an algebraically closed field of characteristic p, but may in fact be any commutative ring. In particular it may be the complex number field $\mathbb{C}$. While for G-modules there is nothing to prove in that case, the results for B-modules are also of interest over fields of characteristic 0.

7.1 The group schemes and the Schubert varieties over the integers

Recall that over an algebraically closed field k we have been considering a connected reductive group G together with a maximal torus T, a Borel group B and embeddings of $\mathrm{SL}(2,k)$ or $\mathrm{PSL}(2,k)$ into G (one for each simple root). Let us assume that G is in fact semi-simple simply connected, so that we are dealing with embeddings $\phi_i : \mathrm{SL}(2,k) \to G$. Now Chevalley and Demazure have shown that corresponding to this data $(G, T, B, \{\phi_i\}_{i\in I})$ over k one gets a group (affine group scheme) $G_{\mathbb{Z}}$ over $\mathbb{Z}$ with subgroups (closed subgroup schemes) $T_{\mathbb{Z}}$, $B_{\mathbb{Z}}$ and embeddings of $\mathrm{SL}(2)_{\mathbb{Z}}$ into $G_{\mathbb{Z}}$, such that the situation over k may be recovered from that over $\mathbb{Z}$ by extension of scalars from $\mathbb{Z}$ to k. One says that $G_{\mathbb{Z}}$ is a $\mathbb{Z}$-form of G. More generally, if S is some structure over k, a $\mathbb{Z}$-form $S_{\mathbb{Z}}$ of S is an analogous structure over $\mathbb{Z}$ together with an isomorphism between S and the structure S_k obtained from $S_{\mathbb{Z}}$ by extension of scalars from $\mathbb{Z}$ to k. The group scheme $\mathrm{SL}(2)_{\mathbb{Z}}$ is the affine algebraic group defined over $\mathbb{Z}$ which represents the functor $R \mapsto \mathrm{SL}(2,R)$. The torus $T_{\mathbb{Z}}$ is diagonalisable. (This means that we are discussing "split" reductive group schemes.) We write $G(R)$ for $G_{\mathbb{Z}}(R)$, the group of points rational over the ring R of the group scheme $G_{\mathbb{Z}}$. For each simple root we get a homomorphism $\phi_i : \mathrm{SL}(2,\mathbb{Z}) \to G(\mathbb{Z})$.

Remark 7.1.1 We do not just try to descend G from k to $\mathbb{Z}$, but G together with B, T and the ϕ_i. That is because G has too many automorphisms, so that there is no canonical "descent" for it. We have "rigidified" by also giving the rest of the data. (Assume the $\mathbb{Z}$-forms $T_{\mathbb{Z}}$ and $\mathrm{SL}(2)_{\mathbb{Z}}$ already chosen.) Thanks to the rigidification we get a *canonical* map from $G(k)$ to the original G.

Remark 7.1.2 Just as one has a $\mathbb{Z}$-form for G, one also has one for G/B. In fact for $(G/B)_{\mathbb{Z}}$ one simply takes $G_{\mathbb{Z}}/B_{\mathbb{Z}}$. It is also straightforward to get analogues over $\mathbb{Z}$ of the Demazure resolutions and one may simply define the Schubert variety $(X_w)_{\mathbb{Z}}$ to be the image of $(P_{s_1} \times^B \cdots \times^B P_{s_m}/B)_{\mathbb{Z}} \to (G/B)_{\mathbb{Z}}$. Unions of Schubert varieties are defined by intersecting their ideal sheafs. It is not obvious, but true, that these constructions do indeed yield $\mathbb{Z}$-forms of Schubert varieties and their unions respectively. In fact, if one looks in [11], one sees that to prove that you really get $\mathbb{Z}$-forms of Schubert varieties, you should first try to understand the $H^0((X_w)_{\mathbb{Z}}, \mathcal{L}^n)$ for high powers $\mathcal{L}^n$ of some ample line bundle $\mathcal{L}$ on $(G/B)_{\mathbb{Z}}$.

7.2 Forms of the Modules

Because of the technicalities indicated in 7.1.2 it is best to avoid the $\mathbb{Z}$-forms of Schubert varieties as much as possible when looking for $\mathbb{Z}$-forms $P(\lambda)_{\mathbb{Z}}$, $Q(\lambda)_{\mathbb{Z}}$ of the B-modules $P(\lambda)$, $Q(\lambda)$. One can then later exploit the understanding of the $P(\lambda)_{\mathbb{Z}}$ to get to grips with the $(X_w)_{\mathbb{Z}}$ and to make the passage to characteristic zero. (Passage to characteristic 0 uses semi-continuity and constructibility properties, cf. [6; 9.2.6.2, 9.4.2, 12.2.4], and generic flatness. See [11; II Chapter 14] and also [17].) Fortunately there is an alternative, thanks to the Demazure resolution. Indeed one knows—but this is also not obvious—that $(X_w)_{\mathbb{Z}}$ is normal, and that leads to the alternative description of $H^0((X_w)_{\mathbb{Z}}, \mathcal{L})$ as being $H^0((Z_i)_{\mathbb{Z}}, \psi_i^*\mathcal{L})$, where $\psi_i : (Z_i)_{\mathbb{Z}} \to$

$(G/B)_{\mathbb{Z}}$ is the Demazure resolution of $(X_w)_{\mathbb{Z}}$. This hopefully explains our clumsy looking constructions below.

Definition 7.2.1 For any $\mu \in X(T)$, let $\mathbb{Z}_\mu$ denote the $B_{\mathbb{Z}}$-module corresponding with the character μ. As a $\mathbb{Z}$-module it is free of rank 1.

Given $\lambda \in X(T)$ we choose simple reflections $s_1, \ldots, s_m$ and anti-dominant λ_1 such that $\lambda = w\lambda_1$, where w has reduced expression $s_1 \cdots s_m$. (We also take m minimal.) Then we define

$$P(\lambda)_{\mathbb{Z}} = \operatorname{ind}_B^{P_1} \operatorname{ind}_B^{P_1} \cdots \operatorname{ind}_B^{P_m} \mathbb{Z}_{\lambda_1},$$

where we have simplified notation a bit by dropping some of the subscripts $\mathbb{Z}$. (Everything is to be done over $\mathbb{Z}$.) We will see later that the notation is justified, by showing that $P(\lambda)_{\mathbb{Z}}$ does not depend on the choices made here. It only depends on λ. Similarly, we define $Q(\lambda)_{\mathbb{Z}}$ inductively:

$$Q(\lambda)_{\mathbb{Z}} = F_1 F_2 \cdots F_m \mathbb{Z}_{\lambda_1},$$

where $F_i(M) := \mathbb{Z}_\rho \otimes_{\mathbb{Z}} \operatorname{ind}_B^{P_i}(\mathbb{Z}_{-s_i\rho} \otimes_{\mathbb{Z}} M)$. The reader will be asked later to check that this is independent of the choices made.

Proposition 7.2.2 (Base change) *For any algebraically closed field k of finite characteristic, $P(\lambda)_k$ is the dual Joseph module of highest weight λ and $Q(\lambda)_k$ is the minimal relative Schubert module of highest weight λ. In other words, $P(\lambda)_{\mathbb{Z}}$ and $Q(\lambda)_{\mathbb{Z}}$ are indeed $\mathbb{Z}$-forms of what the notation suggests.*

Proof: A universal coefficient theorem ([11; I 4.18]) says that we have an exact sequence

$$0 \longrightarrow R^i \operatorname{ind}_{B_{\mathbb{Z}}}^{P_{\mathbb{Z}}}(N) \otimes k \longrightarrow R^i \operatorname{ind}_{B_k}^{P_k}(N_k) \longrightarrow \operatorname{Tor}^{\mathbb{Z}}(R^{i+1} \operatorname{ind}_{B_{\mathbb{Z}}}^{P_{\mathbb{Z}}}(N), k)$$

for any parabolic P and any flat (*i.e.* torsion free) $\mathbb{Z}$-module N with $B_{\mathbb{Z}}$ action. So we can pass to formulas over k whenever the higher derived functors of induction vanish. And they vanish over $\mathbb{Z}$ if they do over all k. (Observe that a finitely generated $\mathbb{Z}$-module M is zero if all M_k vanish.) Thus, from what we know in finite characteristic, we may conclude that, in the notations of 7.2.1, $P(\lambda)_k = \operatorname{ind}_B^{P_1} \operatorname{ind}_B^{P_1} \cdots \operatorname{ind}_B^{P_m} k_{\lambda_1}$. The result for $P(\lambda)_k$ thus follows from Proposition 2.2.5.

For $Q(\lambda)$ we argue similarly. So we must check over k that the higher derived functors of induction vanish at the relevant coefficients and that $k_\rho \otimes \mathrm{ind}_B^{P_s}(k_{-s\rho} \otimes Q(\mu)) = Q(s\mu)$ when s is a simple reflection with $s\mu > \mu$.

First let us consider an example. Take $\mu = -\rho$. Then $Q(-\rho) = k_{-\rho}$ and $P(-s\rho)$ is two-dimensional with weights ρ and $-s\rho$, as the degree of the line bundle $\mathcal{L}(-\rho)$ is 1 on P_s/B. So $Q(-s\rho) = k_{-s\rho}$. In the exact sequence $0 \to Q(-s\rho) \to P(-s\rho) \to P(-\rho) \to 0$ we may interpret $Q(-s\rho)$ as $H^0(P_s/B, \mathcal{I} \otimes \mathcal{L}(-\rho))$ where $\mathcal{I}$ is the ideal sheaf of the point B/B. We claim that $\mathcal{I}$, as a B-equivariant sheaf, is just $\mathcal{L}(-s\rho)[\rho]$. (Notations as in 4.3.14.) Indeed, if one substitutes that for $\mathcal{I}$, one finds $H^0(P_s/B, \mathcal{I} \otimes \mathcal{L}(-\rho)) = k_{-s\rho}$. In view of the classification of B-equivariant sheafs (see Lemma A.4.1), no other equivariant line bundle gives that answer. Of course one may also just compute the action on $\mathcal{I}$ in local co-ordinates.

More generally one thus wants to see that, if $s\mu > \mu$, the evaluation map $\mathrm{ind}_B^{P_s} Q(\mu) \to Q(\mu)$ is surjective and that its kernel $H^0(P_s/B, \mathcal{I} \otimes \mathcal{L}(Q(\mu)))$ equals $Q(s\mu)$. (The surjectivity will yield the necessary vanishing of $H^1(P_s/B, \mathcal{I} \otimes \mathcal{L}(Q(\mu)))$.) Say $\mu = z\lambda_1$, $\lambda_1 \in X(T)^-$, with z minimal. Now if one has a section of $Q(\mu)$, then that is a section of $P(\mu) = H^0(X_z, \mathcal{L}(\lambda_1))$, which extends by zero to ∂X_{sz} by the Mayer–Vietoris Lemma 2.2.11. That section in turn extends to one of $P(s\mu)$ by Ramanathan (Proposition A.2.6), and if one views it as a section of $H^0(P_s \times^B X_z, \mathcal{L}(\lambda_1))$, cf. Proposition 2.2.5, then it vanishes on $H^0(P_s \times^B \partial X_z, \mathcal{L}(\lambda_1))$ by construction. This shows the surjectivity. The kernel of the map $\mathrm{ind}_B^{P_s} Q(\mu) \to Q(\mu)$ consists of sections of $H^0(P_s \times^B X_z, \mathcal{L}(\lambda_1))$ that vanish on $B \times^B X_z \cup P_s \times^B \partial X_z$ and that is just the same as sections of $P(s\mu)$ that vanish on ∂X_{sz}. $\square$

It is worthwhile to make explicit what we have just shown. One may compare it also with Proposition 2.2.15 and 2.3.11.

Lemma 7.2.3 *If $\mu \in X(T)$ and s is a simple reflection such that $s\mu > \mu$, then the following sequence is exact:*

$$0 \longrightarrow Q(s\mu) \longrightarrow H_s(Q(\mu)) \xrightarrow{\text{eval}} Q(\mu) \longrightarrow 0.$$

Exercise 7.2.4 Use the formula $k_\rho \otimes \mathrm{ind}_B^{P_s}(k_{-s\rho} \otimes Q(\mu)) = Q(s\mu)$, valid for $s\mu > \mu$ by the above, to derive a "Demazure character formula" for $Q(\lambda)$, analogous to the one for $P(\lambda)$ in [11; II Proposition 14.18].

Definition 7.2.5 Just like before in Definition 2.3.6 we say that a $B_{\mathbb{Z}}$-module has excellent filtration if it has an exhaustive filtration whose successive filter quotients are isomorphic to direct sums of modules $P(\lambda)_{\mathbb{Z}}$. More generally, if R is any commutative ring we say that a B_R-module has excellent filtration if it has an exhaustive filtration whose successive filter quotients are isomorphic to direct sums of modules $P(\lambda)_R$.

Theorem 7.2.6 *Let $M_{\mathbb{Z}}$ be a $B_{\mathbb{Z}}$-module, finitely generated and flat as a $\mathbb{Z}$-module. Assume that for any algebraically closed field k of finite characteristic the module M_k has excellent filtration. Then so does $M_{\mathbb{Z}}$.*

Proof: First observe that the integers m_λ in $\mathrm{ch}(M_k) = \sum m_\lambda \,\mathrm{ch}(P(\lambda)_k)$ do not depend on the characteristic of k because the $\mathrm{ch}(P(\lambda)_k)$ are linearly independent and do not depend on the characteristic. (They are given by the Demazure character formula, see [11; II Proposition 14.18]. Note that $\mathrm{ch}(P(\lambda)_k) = e^\lambda$ plus terms with weights preceding λ in length–height order.) Fix λ minimal in length–height order with $m_\lambda \neq 0$. Then $\dim_k(\mathrm{Hom}_{B_k}(P(\lambda)_k, M_k)) = m_\lambda$ is independent of the characteristic, so that we expect the injective map $\mathrm{Hom}_{B_{\mathbb{Z}}}(P(\lambda)_{\mathbb{Z}}, M_{\mathbb{Z}})) \otimes k \to \mathrm{Hom}_{B_k}(P(\lambda)_k, M_k)$ to be an isomorphism. To see this is indeed so, recall the corresponding universal coefficient theorem ([11; I 4.18]) which says that we have an exact sequence

$$0 \longrightarrow H^i(B_{\mathbb{Z}}, N) \otimes k \longrightarrow H^i(B_k, N_k) \longrightarrow \mathrm{Tor}^{\mathbb{Z}}(H^{i+1}(B_{\mathbb{Z}}, N), k)$$

for any any flat (*i.e.* torsion free) $\mathbb{Z}$-module N with $B_{\mathbb{Z}}$ action.

So we wish to get hold of the $\mathbb{Z}$-module $H^i(B_{\mathbb{Z}}, N)$, with $N = \mathrm{Hom}_{\mathbb{Z}}(P(\lambda)_{\mathbb{Z}}, M_{\mathbb{Z}})$. It is finitely generated by weight considerations as in [11; II Prop. 4.10]. (The weight spaces of the U-cohomology are finitely generated.) Now $H^i(B_k, \mathrm{Hom}_{\mathbb{Z}}(P(\lambda)_{\mathbb{Z}}, M_{\mathbb{Z}}) \otimes k) = H^i(B_k, \mathrm{Hom}_k(P(\lambda)_k, M_k)) = \mathrm{Ext}^i_{B_k}(P(\lambda)_k, M_k)$ vanishes for $i > 0$ by the strong form of Polo's theorem.

Next we consider the natural homomorphism

$$\phi : P(\lambda)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{Hom}_{B_{\mathbb{Z}}}(P(\lambda)_{\mathbb{Z}}, M_{\mathbb{Z}})) \longrightarrow M_{\mathbb{Z}}.$$

When tensored with k one always gets an isomorphism from a direct sum of m_λ copies of $P(\lambda)_k$ with a submodule of M_k. By the elementary divisors theorem this means the cokernel of ϕ is torsion free and thus is a module as in the theorem, but with smaller rank. The theorem follows by induction on the rank. □

Corollary 7.2.7 (Uniqueness) *Let $M_{\mathbb{Z}}$ be a $B_{\mathbb{Z}}$-module, finitely generated and flat as a $\mathbb{Z}$-module. Assume that for any algebraically closed field k of finite characteristic the module M_k is the dual Joseph module of highest weight λ. Then $M_{\mathbb{Z}}$ is isomorphic with $P(\lambda)_{\mathbb{Z}}$.*

Proof: In the excellent filtration of $M_{\mathbb{Z}}$ we must find $P(\lambda)_{\mathbb{Z}}$, and nothing else, because of characters. (Compare the proof of the preceding theorem.) Note that it follows that the choices made in the construction of $P(\lambda)_{\mathbb{Z}}$ do not make a difference. □

Exercise 7.2.8 (i) Show that $\mathrm{Ext}_B(Q(\lambda), Q(\mu))$ vanishes when $\lambda = \mu$ and also when $-\lambda$ precedes $-\mu$ in length–height order.

(ii) Now formulate and prove a similar theorem and corollary with relative Schubert filtrations.

Theorem 7.2.9 (Main Theorem; Mathieu [20]) *Let R be a commutative ring, M a B_R-module with excellent filtration. Let $\lambda \in X(T)^-$. Then $\lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} M$ has excellent filtration.*

Proof: As the $P(\mu)_{\mathbb{Z}}$ are flat, it suffices to take $R = \mathbb{Z}$. By the Local–Global Theorem 7.2.6 it now follows from Polo's Conjecture 6.2.1 as proved in the previous chapter. □

In the same vein we get

Theorem 7.2.10 (Restriction Theorem) *Let R be a commutative ring, M a B_R-module with excellent filtration. Let L_R be the Levi factor of a parabolic, corresponding with a subset of the simple roots. Then $\mathrm{res}^{B_R}_{L_R \cap B_R} M$ is an $L_R \cap B_R$-module with excellent filtration.* □

7.3 Passage to Characteristic 0

Many properties that have been proved with the help of Frobenius splittings easily extend to characteristic 0 by semi-continuity and constructibility properties as developed in [6]. We will illustrate this with an example. Observe however that in characteristic 0 our theory says nothing interesting about G-modules because of complete reducibility. On the other hand, the main theorem certainly gives non-obvious results for B-modules. We do not even know a direct proof that, for anti-dominant λ and dominant μ, the character of $\lambda \otimes P(\mu)$ is a sum of characters of dual Joseph modules.

We know in finite characteristic that Schubert varieties are normal. As is well known this yields:

Lemma 7.3.1 *Over the complex numbers Schubert varieties are also normal.*

Proof: Let $w \in W$. Let $(X_w)_{\mathbb{Z}}$ be defined as the closure of $B_{\mathbb{Z}}wB_{\mathbb{Z}}/B_{\mathbb{Z}}$ in $G_{\mathbb{Z}}/B_{\mathbb{Z}}$. In other words, the ideal sheaf of $(X_w)_{\mathbb{Z}}$ consists of the functions that pull back to zero on $B_{\mathbb{Z}}wB_{\mathbb{Z}}$. It is clear that $(X_w)_{\mathbb{Z}}$ is flat over $\mathbb{Z}$. (We do not really need that much; generic flatness would have been enough.) Now $(X_w)_{\mathbb{C}}$ is obtained by flat extension, and one sees it is the Schubert variety we want to study. It is reduced, connected, irreducible of dimension $l(w)$ and it contains BwB/B. So by [6; 9.2.6.2, 12.2.4] and common sense (for the containment), there is a neighborhood of the generic point of Spec($\mathbb{Z}$), such that the analogous properties hold for $(X_w)_k$ whenever k is a geometric point of V. (That is, k is algebraically closed and its image in Spec($\mathbb{Z}$) lies in V.) But then for such a geometric point of finite characteristic, $(X_w)_k$ cannot be anything else than a Schubert variety. So it is normal. Now the same Theorem [6; 12.2.4] finishes the job. □

Lemma 7.3.2 *The $B_{\mathbb{C}}$-module $P(\lambda)_{\mathbb{C}}$ is indeed $H^0(X_w, \mathcal{L}_{\lambda_1})$, with $w \in W$ and λ_1 anti-dominant such that $\lambda = w\lambda_1$.*

Proof: As $\mathbb{C}$ is flat over $\mathbb{Z}$, we have $P(\lambda)_{\mathbb{C}} = \operatorname{ind}_B^{P_1} \operatorname{ind}_B^{P_2} \cdots \operatorname{ind}_B^{P_m} \mathbb{C}_{\lambda_1}$. So what we need is the analogue of Proposition 2.2.5. But that depended on normality of Schubert varieties, so it goes through. □

Appendix A

Geometry

In this appendix we give a more extensive discussion of Frobenius splitting of varieties. Further we tie up some loose ends that have more to do with algebraic geometry than with B-modules.

The notion of Frobenius split varieties was introduced by V. Mehta and A. Ramanathan in 1984. We refer the reader to [32] for historical remarks. Indeed, much of the material in this appendix is copied from this source.

A.1 Frobenius Splitting of Varieties

In this section and the next some proofs are sketchy or absent. For more information see [32], [24], [31]. Let k be a algebraically closed field of characteristic $p > 0$. Let A be any k-algebra. In this situation, we have the Frobenius ring homomorphism $a \mapsto a^p$ of A. For a variety X over k we have the absolute Frobenius morphism $F : X \to X$ which is induced by the Frobenius ring homomorphism on any of its affine open subsets. Note that the map F is identity on the underlying topological space of X and on functions it is the pth power map. By abuse of notation, we also use F to denote the pth power map $F : \mathcal{O}_X \to F_*\mathcal{O}_X$. If $\mathcal{G}$ is a coherent sheaf on X then the direct image $F_*\mathcal{G}$ is the same as $\mathcal{G}$ as a sheaf of abelian groups; only its $\mathcal{O}_X$-module structure $\circ$ is via the Frobenius morphism, *i.e.* $f \circ g = f^p g$, for $f \in \mathcal{O}_X$ and $g \in F_*\mathcal{G}$.

Definition A.1.1 1. A variety X over k is called Frobenius split if the pth power map $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ has a splitting *i.e.* an $\mathcal{O}_X$-module morphism

$\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that the composite $\phi F : \mathcal{O}_X \to \mathcal{O}_X$ is identity.

2. If Y is a closed subvariety of X with the ideal sheaf $\mathcal{I}$ such that $\phi(F_*\mathcal{I}) = \mathcal{I}$ then we say Y is compatibly split in X.

3. If $Y_1, \ldots, Y_n$ are closed subvarieties which are all compatibly split by the same Frobenius splitting of X then we say that the closed subvarieties $Y_1, \ldots, Y_n$ are *simultaneously compatibly split* in X.

Exercise A.1.2 Check that these definitions agree with those given earlier in 4.3.

The following remark was used by Ramanathan to study the scheme theoretic intersection of two unions of Schubert varieties (cf. proof of Mayer–Vietoris Lemma 2.2.11).

Remark A.1.3 If X is a scheme and $F : X \to X$ has a splitting then X is necessarily reduced. This is a consequence of the fact that the Frobenius morphism is the pth power map on functions and if the scheme is Frobenius split then this map is an injection.

A Frobenius splitting of a variety X is thus an element in the set of global sections $H^0(X, (F_*\mathcal{O}_X)^*)$ of the dual of $F_*\mathcal{O}_X$. Let us assume now that X is a smooth variety of dimension n. Let ω_X be its canonical bundle. Using duality theory—an alternative will be discussed in section A.3—we see that

$$\begin{aligned} H^0(X, (F_*\mathcal{O}_X)^*) &= H^n(X, F_*\mathcal{O}_X \otimes \omega_X) \\ &= H^n(X, F_*(\mathcal{O}_X \otimes F^*\omega_X)) \\ &= F_*H^n(X, \omega_X^p) \\ &= H^0(X, \omega_X^{1-p}). \end{aligned}$$

The following proposition tells that a normal variety will be Frobenius split if one of its desingularisation is Frobenius split.

Remark A.1.4 Conversely, there are proofs of normality based on Frobenius splittings, using Proposition A.5.2. See [25].

Proposition A.1.5 *Let $f : Z \to X$ be a morphism of algebraic varieties. Assume that $f_*\mathcal{O}_Z = \mathcal{O}_X$. (We will say that f has the direct image property.) Then,*

(i) *If Z is Frobenius split then X is also Frobenius split.*

(ii) *If Y is a closed subvariety of Z which is compatibly split in Z then its image $f(Y)$ is compatibly split in X.*

Proof: (i) For an open subset U of X the splitting gives an element of $\mathrm{End}_F(\mathcal{O}_Z(f^{-1}(U)))$ that sends the function 1 to itself.

(ii) Let $\mathcal{I} \subset \mathcal{O}_Z$ be the ideal sheaf of Y. Then as $f_*\mathcal{O}_Z = \mathcal{O}_X$, the ideal sheaf of $f(Y)$ is $f_*\mathcal{I}$. Now it is an easy exercise to see that the "pushed" splitting of X splits $f(Y)$. □

Lemma A.1.6 *If a splitting of the variety X is compatible with the subvarieties Y_1 and Y_2 then it is also compatible with $Y_1 \cap Y_2$ and $Y_1 \cup Y_2$.*
It is compatible with a subvariety Y if and only if it is compatible with each irreducible component of Y.

Proof: For the first part one uses that $\mathcal{I}_{Y_1 \cap Y_2} = \mathcal{I}_{Y_1} + \mathcal{I}_{Y_2}$ and $\mathcal{I}_{Y_1 \cup Y_2} = \mathcal{I}_{Y_1} \cap \mathcal{I}_{Y_2}$. For the second one shows that a splitting σ is compatible with a subvariety Z if and only if there is an open subset U such that $U \cap Z$ is dense in Z and such that $\sigma|_U$ is compatible with $U \cap Z$. □

Now we give a criterion for a section of ω_X^{1-p} of a smooth variety to be a splitting.

Proposition A.1.7 *Let Z be a smooth projective variety of dimension n. Let $Z_1, \ldots, Z_n$ be smooth irreducible subvarieties of codimension 1 such that the scheme theoretic intersection $Z_{i_1} \cap \cdots \cap Z_{i_r}$ is smooth irreducible and of dimension $n - r$ for all $1 \leq i_1 < \cdots < i_r \leq n$. If there exists a section $s \in H^0(Z, \omega_X^{-1})$ such that $\mathrm{div}(s)$, the divisor of zeroes of s, is $Z_1 + \cdots + Z_n + D$ where D is an effective divisor not passing through the point $P = Z_1 \cap \cdots \cap Z_n$ then the section $\sigma = s^{p-1}$ of ω_X^{1-p} gives, by duality, a splitting of Z (or a non-zero multiple of one) which makes all the intersections $Z_{i_1} \cap \cdots \cap Z_{i_r}$ compatibly split.*

Note that an element σ of $\mathrm{End}_F(X)$ is a splitting if and only if $\sigma(1) = 1$. If X is projective, then in any case $\sigma(1)$ is a global function, hence constant. Thus it suffices to check its value at a single point. In the case of the proposition one uses the point P and makes a computation in local coordinates.

Let G be a connected simply connected semi-simple algebraic group over k. (Or let it be as in 2.2.8.) Let T be a maximal torus, $B \supset T$ a Borel subgroup and $W = N(T)/T$ the Weyl group of G. Let $w_0 \in W$ denote the longest element of the Weyl group.

The homogeneous space G/B is a projective variety. A closure of a B-orbit in G/B is called a Schubert variety. The B-orbits in G/B are indexed in a natural way by elements of W. If $P \supset B$ is a parabolic subgroup of G, then there are only finitely many B-orbits in the projective variety G/P. We refer the reader to Kempf's paper ([14]), for basic facts about the geometry of Schubert varieties.

Let D denote the divisor sum of all codimension one Schubert varieties of G/B. Let $\widetilde{D}$ denote the sum of w_0 translates of codimension one Schubert varieties. Then the divisor $D + \widetilde{D}$ gives the anti-canonical bundle $\omega_{G/B}^{-1}$ of G/B. It is the image of a divisor in a Demazure resolution that satisfies the criterion A.1.7 for a splitting and by pushing forward with Lemma A.1.5 one gets a splitting which simultaneously splits all the Schubert varieties of G/B. Therefore we have the following theorem.

Theorem A.1.8 *Let G be connected simply connected semi-simple algebraic group. Let P be a parabolic subgroup of G. Then the projective variety G/P is Frobenius split. Further, all the Schubert varieties of G/P are simultaneously compatibly split.*

Proof: One uses Lemma A.1.6 to deal with Schubert varieties of higher codimension. □

Theorem A.1.9 1. *The product $G/B \times G/B$ is Frobenius split. Further the diagonal $\Delta = \{(x, x) |\, x \in G/B\}$ is compatibly split in $G/B \times G/B$.*

2. *The variety $G \times^B (G/B \times G/B)$ is Frobenius split. Further all the double Schubert varieties are simultaneously compatibly split.*

This will be proved below (Propositions A.4.9 and A.4.8).

A.2 Applications of Frobenius Splitting

In this section we prove certain vanishing theorems for the Frobenius split variety X.

First some remarks on the direct and inverse images of sheaves under the absolute Frobenius morphism F. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules on X. Recall that the direct image sheaf $F_*\mathcal{M}$ is the same as $\mathcal{M}$ as a sheaf of abelian groups, but the $\mathcal{O}_X$-module structure is changed to $f \circ m = f^p m$, for $f \in \mathcal{O}_X$ and $m \in \mathcal{M}$. As a way of notation, we will identify $\mathcal{M}$ and $F_*\mathcal{M}$ as sets. The pullback $F^*\mathcal{M}$ is by definition $\mathcal{M} \otimes_{\mathcal{O}_X} F_*\mathcal{O}'_X$. Here the prime has been put in to denote that the $\mathcal{O}_X$-module structure is given by the usual multiplication on the second factor, *i.e.* $f(m \otimes g) = fm \otimes g = m \otimes fg$ (and not $m \otimes f^p g$). The sheaf $\mathcal{M} \otimes F_*\mathcal{O}_X$ with its $\mathcal{O}_X$-module structure coming from $\mathcal{M}$, *i.e.* $f(m \otimes g) = fm \otimes g = m \otimes f^p g$, is by definition $F_*F^*\mathcal{M}$. This gives us the projection formula: $F_*F^*\mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X$.

If we consider a line bundle $\mathcal{L}$ on X, we get a natural isomorphism $F^*\mathcal{L} \approx \mathcal{L}^p$. Tensoring the Frobenius exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X \longrightarrow \mathcal{C} \longrightarrow 0$$

by $\mathcal{L}$ and taking the cohomology, we get a natural map

$$H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L} \otimes F_*\mathcal{O}_X) = H^i(X, F_*F^*\mathcal{L}) = H^i(X, F_*\mathcal{L}^p).$$

Proposition A.2.1 *Let X be a projective variety which is Frobenius split. Let Y be a closed subvariety of X which is compatibly split. Let $\mathcal{L}$ be a line bundle on X such that $H^i(X, \mathcal{L}^m) = H^i(Y, \mathcal{L}^m) = 0$ for some i and for all large m. Then $H^i(X, \mathcal{L}) = 0 = H^i(Y, \mathcal{L})$.*

Proof: We have a natural map $H^i(X, \mathcal{L}) \to H^i(X, F_*\mathcal{L}^p)$. Further as F is affine (*i.e.* inverse image of an affine open set is affine), it commutes with the cohomology. Thus $H^i(X, F_*\mathcal{L}^p) = H^i(X, \mathcal{L}^p)$. Now as the sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes F_*\mathcal{O}_X \longrightarrow \mathcal{L} \otimes \mathcal{C} \longrightarrow 0$$

is split exact this morphism is injective. Therefore, by iteration, we have an injective morphism $H^i(X, \mathcal{L}) \to H^i(X, \mathcal{L}^{p^\nu})$ for all ν. Thus $H^i(X, \mathcal{L}^{p\nu}) = 0$ implies that $H^i(X, \mathcal{L}) = 0 = H^i(Y, \mathcal{L})$. $\square$

The above proposition together with Serre vanishing theorem gives us the following corollary.

Corollary A.2.2 *Let $\mathcal{L}$ be an ample line bundle on X. If X is Frobenius split, then $H^i(X, \mathcal{L}) = 0$ for all $i > 0$. Further, if $Y \subset X$ is compatibly split, $H^i(Y, \mathcal{L}) = 0$ and the restriction map $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L})$ is surjective.*

Proof: To see the surjectivity of the restriction map, we consider

$$\begin{array}{ccc} H^0(X, \mathcal{L}) & \longrightarrow & H^0(X, \mathcal{L}^{p^\nu}) \\ \downarrow & & \downarrow \\ H^0(Y, \mathcal{L}) & \longrightarrow & H^0(Y, \mathcal{L}^{p^\nu}) \end{array}$$

As the horizontal arrows are split, it is enough to see the surjectivity of the global sections for a high power of $\mathcal{L}$. Thus the result. □

For Schubert varieties Ramanathan proved something better than what one can achieve with the above. He also deals with base point free line bundles on G/B that are not ample. So he deals with the $\mathcal{L}(\lambda)$ with λ anti-dominant, but not regular anti-dominant. We need this stronger result. Therefore let us now discuss a more refined notion of splitting (although we have no other application than this stronger result of Ramanathan).

Definition A.2.3 Let $\mathcal{L}$ be a line bundle on X and $s : \mathcal{O}_X \to \mathcal{L}$ a non-zero section of $\mathcal{L}$ with zeroes precisely on D.

1. We say X is Frobenius D-split (or less precisely Frobenius $\mathcal{L}$-split) if there exists $\psi : F_*\mathcal{L} \to \mathcal{O}_X$ such that the composite $\phi = \psi F_*(s)$

$$\begin{array}{ccc} F_*\mathcal{O}_X & \xrightarrow{\phi} & \mathcal{O}_X \\ {\scriptstyle F_*(s)}\searrow & & \nearrow{\scriptstyle \psi} \\ & F_*\mathcal{L} & \end{array}$$

is a Frobenius splitting of X.

2. If Y is a closed subvariety of X such that

(i) no irreducible component of Y is contained in the support $\operatorname{supp} D$,

(ii) ϕ gives a compatible splitting of Y in X,

then we say Y is compatibly D-split in X.

3. If all subvarieties $Y_1, \ldots, Y_r$ are compatibly D-split by the same D-splitting of X then we say that $Y_1, \ldots, Y_r$ are simultaneously compatibly D-split in X.

Remark A.2.4 1. We note that if X is Frobenius split, it is also ω_X^{1-p}-split, as any section which gives a splitting vanishes on a divisor whose associated line bundle is ω_X^{1-p}.

2. Let D' be another Cartier divisor with $0 \leq D' \leq D$. Then if X is D-split it is also D'-split.

We now see a consequence of D-splittings.

Proposition A.2.5 *If X is $\mathcal{L}$-split with $\mathcal{L}$ ample and $\mathcal{M}$ is a line bundle without base points (i.e. for every $x \in X$, there exists $s \in H^0(X, \mathcal{M})$ such that $s(x) \neq 0$) then $H^i(X, \mathcal{M}) = 0$ for $i > 0$. If further Y is compatibly $\mathcal{L}$-split then $H^i(Y, \mathcal{M}) = 0$ for $i > 0$ and the restriction map $H^0(X, \mathcal{M}) \to H^0(Y, \mathcal{M})$ is surjective.*

For the proof we refer the reader to Ramanathan [32].

Let us now consider the case when X is the projective homogeneous space G/B. In this case the divisor $(p-1)(D+\widetilde{D})$ gives a splitting. The line bundle corresponding to the divisor D is ample, in fact it is the line bundle given by the character $-\rho$. Also as G/B is homogeneous, any homogeneous line bundle with a non-zero section is base point free. Therefore we get the following proposition.

Proposition A.2.6 (Ramanathan, [31; Theorem 3]) *Let $\mathcal{L}$ be a line bundle on G/B such that $H^0(G/B, \mathcal{L}) \neq 0$. Then $H^i(X, \mathcal{L}) = 0$ for any union of Schubert varieties X and for all $i > 0$. Further the restriction map $H^i(G/B, \mathcal{L}) \to H^i(X, \mathcal{L})$ is surjective for all i.*

Remark A.2.7 The case $X = G/B$ is known as Kempf's vanishing theorem.

Remark A.2.8 Let S be a union of product of Schubert varieties. Consider the fibration

$$\begin{array}{c} G \times^B S \\ \downarrow \pi \\ G/B \end{array}$$

It is locally trivial in the Zariski topology (Exercise 1.2.1), so the structure sheaf of $G \times^B S$ is certainly flat over the base G/B. The proposition above gives us that $R^i\pi_*\mathcal{O} = 0$ for $i > 0$ because $R^i\pi_*\mathcal{O}$ is a vector bundle on G/B with fibre isomorphic with $H^i(S, \mathcal{O})$ which vanishes as $\mathcal{O}$ is base point free on G/B. (Use [7; Grauert's corollary to Semicontinuity].) Similarly, if P, Q are parabolics and $\mathcal{L}$ is an ample line bundle (or one without base points) on $G \times^B (G/P \times G/Q)$ then for any union S of products of Schubert varieties in $G/P \times G/Q$ the higher $R^i f_*(\mathcal{L}|_{G\times^B S})$ vanish, where $f : G \times^B S \to G/B$. So $H^i(G/B, f_*(\mathcal{L}|_{G\times^B S})) = H^i(G \times^B S, \mathcal{L}|_{G\times^B S})$ by Leray ([7; III, Ex. 8.1]).

Exercise A.2.9 Let P be a parabolic and let S be a union of Schubert varieties in G/P. Argue as in the remark above to show that if $\mathcal{L}$ is a line bundle on G/P, then $H^i(S, \mathcal{L}) = H^i(\pi^{-1}(S), \pi^*\mathcal{L})$, with $\pi : G/B \to G/P$. Next assume $\mathcal{L}$ is base point free and let S_1 be a union of Schubert varieties in G/B with $\pi(S_1) = S$. Show that $H^0(S_1, \pi^*\mathcal{L}) = H^0(S, \mathcal{L})$.

A.3 Cartier Operators and Splittings

We now give another approach to the isomorphism $\mathcal{E}nd_F(X) \approx \omega_X^{1-p}$. It does not make reference to duality theory, but only to the Cartier operator. With this description it will be quite feasible to make explicit computations with splittings in local coordinates, if the splittings are given as sections of ω_X^{1-p}.

Let X be a variety of dimension n over k, with k algebraically closed of characteristic p, as usual. We consider the DeRham complex

$$0 \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \Omega_X^n \longrightarrow 0$$

with as differential d the usual exterior differentiation. Because this differential is not $\mathcal{O}_X$-linear, we twist the $\mathcal{O}_X$-module structure on Ω_X^i by putting $f * \omega = f^p\omega$ for a section $f \in H^0(U, \mathcal{O}_X)$ and a differential i-form $\omega \in H^0(U, \Omega_X^i)$. With this twisted module structure the DeRham complex is a complex of coherent $\mathcal{O}_X$-modules, and the exterior algebra $\Omega_X^* = \bigoplus_{i=0}^n \Omega_X^i$ is a differential graded $\mathcal{O}_X$-algebra. We denote its cohomology sheafs $\mathcal{H}_{\mathrm{dR}}^i$. So if U is an affine open subset, then $H^0(U, \mathcal{H}_{\mathrm{dR}}^i)$ consists of all closed differential i-forms on U modulo the exact ones. Now consider the map $\gamma : f \mapsto$ class of $f^{p-1}df$ from $\mathcal{O}_X$ to $\mathcal{H}_{\mathrm{dR}}^1$.

Lemma A.3.1 *γ is a derivation and thus induces an $\mathcal{O}_X$-algebra homomorphism $c : \Omega_X^* \to \mathcal{H}_{\mathrm{dR}}^*$.*

Remark A.3.2 Note that one should put the ordinary $\mathcal{O}_X$-module structure on Ω_X^* here, not the twisted one that is used for $\mathcal{H}_{\mathrm{dR}}^*$.

Proof of Lemma A.3.1: With

$$\Phi(X, Y) = ((X + Y)^p - X^p - Y^p)/p \in \mathbb{Z}[X, Y]$$

we get

$$\begin{aligned}(f + g)^{p-1}d(f + g) &= f^{p-1}df + g^{p-1}dg + d\Phi(f, g)\\ (fg)^{p-1}d(fg) &= g * f^{p-1}df + f * g^{p-1}dg,\end{aligned}$$

where the first equality is a consequence of the fact that

$$p(X + Y)^{p-1}d(X + Y) = pX^{p-1}dX + pY^{p-1}dY + pd\Phi(X, Y)$$

in the torsion free $\mathbb{Z}$-module $\Omega^1_{\mathbb{Z}[X,Y]}$. □

Proposition A.3.3 *If X is smooth, the homomorphism c is bijective. The inverse map $C : \mathcal{H}_{\mathrm{dR}}^* \to \Omega_X^*$ is called the Cartier operator (cf. [26]).*

Proof: To check that a map of coherent sheafs is an isomorphism it suffices to check that one gets an isomorphism after passing to the completion at an arbitrary closed point. But then we are simply dealing with the DeRham complex for a power series ring in n variables over k and everything can be made very explicit (exercise). □

Remark A.3.4 Here are some formulas satisfied by the Cartier operator, in sloppy notation. In view of these formulas the connection with Frobenius splittings is not surprising.

(i) $C(f^p\tau) = fC(\tau)$

(ii) $C(d\tau) = 0$

(iii) $C(\operatorname{dlog} f) = \operatorname{dlog} f$, where $\operatorname{dlog} f$ stands for $(1/f)df$ if f is invertible (or inverted).

(iv) $C(\xi \wedge \tau) = C(\xi) \wedge C(\tau)$

Here f is a function and ξ, τ are forms.

Proposition A.3.5 *If X is smooth, we have a natural isomorphism*

$$\mathcal{E}nd_F(X) \approx \omega_X^{1-p} = \mathcal{H}om(\omega_X^p, \omega_X),$$

where ω_X is the canonical line bundle Ω_X^n. If τ is a local generator of ω_X, f a local section of $\mathcal{O}_X$, ϕ a local homomorphism $\omega_X^p \to \omega_X$, then the corresponding local section σ of $\mathcal{E}nd_F(X)$ is defined by $\sigma(f)\tau = C(\text{class of } \phi(f\tau^{\otimes p}))$.

Proof: One checks that $C(\text{class of } \phi(f\tau^{\otimes p}))/\tau$ does not depend on the choice of τ, so that σ depends only on ϕ. To see that the map $\phi \mapsto \sigma$ defines an isomorphism of line bundles we may argue as in the previous proof. □

A.4 Canonical splitting of the Demazure resolution

We wish to study $\mathcal{E}nd_F(Z_n, D_n)$ for an arbitrary sequence $s_{i_1}, \ldots s_{i_n}$ of simple reflections. In particular, we wish to prove Proposition 4.3.15. We start with the problem of recognizing B-equivariant bundles on Z_n.

Lemma A.4.1 *Let X be a connected projective variety with B action and x an invariant point. Let $\mathcal{E}$, $\mathcal{F}$ be B-equivariant line bundles that are isomorphic as line bundles. If their fibres over x are B-equivariantly isomorphic, then the line bundles themselves are B-equivariantly isomorphic.*

Proof: Tensoring with $\mathcal{E}^*$ we reduce to the case that $\mathcal{E}$, $\mathcal{F}$ are trivial as line bundles. Then

$$H^0(X, \mathcal{E}) \approx H^0(\{x\}, \mathcal{E}) \approx H^0(\{x\}, \mathcal{F}) \approx H^0(X, \mathcal{F})$$

equivariantly. But the global sections generate a trivial sheaf everywhere, so the B action on such a sheaf is determined by what it does on global sections. □

This lemma takes care of recognizing the B action, so let us now look at the Picard group of Z_n.

Lemma A.4.2 *The isomorphism type of a line bundle on Z_n is determined exactly by the degrees of the restrictions to the n embedded $\mathbb{P}^1$'s of the form $B \times^B \cdots \times^B P_i \times^B \cdots \times^B B/B$.*

Proof: This is clear for $Z_{n,n} \approx \mathbb{P}^1$, so we work our way back to Z_n by means of the fibrations $\pi_j : Z_{j,n} = P_j \times^B P_{j+1} \cdots P_n/B \to P_j/B \approx \mathbb{P}^1$ with fibre $Z_{j+1,n}$. Use [7; Ch. II, Prop. 6.5] with as divisor the fibre of the point "at infinity" s_{i_j} of P_j/B and observe that the complement of this fibre is a direct product of $Z_{j+1,n}$ with an affine line. Apply [7; Ch. II, Prop. 6.6] to this complement. □

Corollary A.4.3 *Under the standing hypothesis 2.2.8 all line bundles on G/B come from G-equivariant ones. The equivariant structure is unique up to a twist by a character of G. In particular, if G is its own commutator subgroup then the equivariant structure is unique.*

Proof: The regular representation of G restricts to a faithful representation of its commutator group, so the fundamental weights of the commutator group are restrictions of weights of B. Therefore the set of degrees of restrictions to the projective lines P_s/B ($s \in S$) runs through all possibilities as we vary the line bundle over all $\mathcal{L}(\lambda)$, $\lambda \in X(T)$. And a line bundle is clearly determined by its pullback to a Demazure resolution of G/B. To finish, argue as in the proof of Lemma A.4.1. □

Exercise A.4.4 Let P be a parabolic and X a space with B action. Show that every P-equivariant vector bundle on $P \times^B X$ is of the form $P \times^B \mathcal{V}$ with $\mathcal{V}$ a B-equivariant bundle on X.

The following lemma may be used to pass between $\mathcal{E}nd_F(Z_n, D_n)$ and $\mathcal{E}nd_F(Z_n) \otimes \mathcal{I}_{D_n}^{p-1}$.

Lemma A.4.5 *Let A be a domain of characteristic p and (f) a principal ideal in it. Then* $\mathrm{End}_F(A,(f)) = (f)^{p-1} * \mathrm{End}_F(A)$.

Proof: That the left-hand side contains the right-hand side is clear. Let $\sigma \in \mathrm{End}_F(A,(f))$. Then $\sigma(fa) = f\tau(a)$ defines a map τ from A to itself. One checks that $\tau \in \mathrm{End}_F(A)$ and that $f\sigma = f(f^{p-1} * \tau)$. □

Proposition A.4.6 *The sheaf $\mathcal{E}nd_F(Z_n, D_n)$ is B-equivariantly isomorphic with $\varphi^*\mathcal{L}((1-p)\rho)[(p-1)\rho]$, so that if $\varphi : Z_n \to G/B$ is surjective,* $\mathrm{End}_F(Z_n, D_n)$ *is B-equivariantly isomorphic with* $k_{(p-1)\rho} \otimes H^0(G/B, \mathcal{L}((1-p)\rho))$.

Proof: By Lemmas A.4.5 and A.3.5 all we have to show for the first statement is that $\omega_{Z_n}(-D_n) \cong \varphi^*\mathcal{L}(\rho)[-\rho]$, equivariantly. We argue again by induction, using the fibration $\pi_j : Z_{j,n} = P_j \times^B P_{j+1} \cdots P_n/B \to P_j/B \approx \mathbb{P}^1$ with fibre $Z_{j+1,n}$. Let $D_{j,n}$ denote the analogue of D_n in $Z_{j,n}$. Thus $D_{j,n}$ is a divisor with $n-j+1$ components intersecting in a point x. The required result is easy for $j = n$. Indeed if α is the simple root corresponding with P_n, one gets a local coordinate t on $P_n/B \approx \mathbb{P}^1$ from $t \mapsto x_{-\alpha}(t)B/B$ and the stalk at the "origin" x of $\omega_{Z_{n,n}}(-D_{n,n})$ is generated by dt/t on which T acts trivially. Further the degree of the line bundle is -1, so by our recognition Lemma A.4.1 we must have $\omega_{Z_{n,n}}(-D_{n,n}) \cong \varphi_{n,n}^*\mathcal{L}(\rho)[-\rho]$, equivariantly.

Now assume such a result for $\omega_{Z_{j+1,n}}(-D_{j+1,n})$ and consider $\omega_{Z_{j,n}}(-D_{j,n})$. It is the tensor product of two line bundles. The first one, say $\mathcal{R}$, is the is the relative canonical bundle $\omega_{Z_{j,n}/\mathbb{P}^1} = \bigwedge^{n-j} \Omega_{j,n/\mathbb{P}^1}$, twisted by $\mathcal{L}(P_j \times^B D_{j+1,n})$. The second is the pullback of $\omega_{\mathbb{P}^1}(-\{x\})$, with x "as above". Let us study $\mathcal{R}$ through its restrictions to the various copies of $\mathbb{P}^1$, cf. Lemmas A.4.1 and A.4.2. By base change for relative differentials, see [7; II, 8.2], the restriction of $\omega_{Z_{j,n}/\mathbb{P}^1}$ to $B \times^B Z_{j+1,n}$ is just $\omega_{Z_{j+1,n}}$. So $\mathcal{R}$ restricts to $\omega_{Z_{j+1,n}}(-D_{j+1,n})$, which we know. We also need the restriction of $\mathcal{R}$ to P_j/B. Now that is a P_j-equivariant sheaf whose fibre at x has trivial T action, so it must be the structure sheaf on P_j/B. The sheaf $\omega_{\mathbb{P}^1}(-\{x\})$ we have already found to be the pullback from G/B of $\mathcal{L}(\rho)[-\rho]$, and its pullback to $Z_{j,n}$ is easy to understand in terms of its restrictions to the relevant $\mathbb{P}^1$'s. So we have all the ingredients to conclude $\omega_{Z_{j,n}}(-D_{j,n}) \cong \varphi^*\mathcal{L}(\rho)[-\rho]$, equivariantly. To prove the last statement of the proposition, use Exercise A.4.4 and the fibrations π_j to see that $H^0(\varphi^*\mathcal{L}((1-p)\rho)) = H_{s_1} \circ \cdots \circ H_{s_n}((1-p)\rho)$. □

Proposition A.4.7 (Proposition 4.3.17) *There exists $\sigma \in \mathrm{End}_F(Z_n, D_n)$ which is a canonical splitting.*

Proof: We have already described in 4.3.17 how one proves this with the criterion A.1.7. Let us tell it a little differently now. Let $s \in H^0(G/B, \mathcal{L}((1-p)\rho)[(p-1)\rho])$ be a weight vector of weight zero. It does not vanish at B/B. We wish to show that its pullback defines $\sigma_n \in \mathrm{End}_F(Z_n, D_n)$ with $\sigma_n(1) \neq 0$. As Z_n is complete, $\sigma_n(1)$ is a constant function. Call the constant c_n. We argue by induction, the case $n = 0$ being easy. Now an exercise in chasing duality, say with the Cartier operator, shows that the restriction of $\sigma_{j,n}(1)$ to $Z_{j+1,n}$ is just $\sigma_{j+1,n}(1)$ in hopefully self-explanatory notation. (Use reasonable identifications, choose a local coordinate t on P_j/B which vanishes at B/B and use that the fibration $Z_{j,n} \to P_j/B$ is trivial in a neighborhood of B/B.) So $c_{j,n} = c_{j+1,n}$, which is non-zero by inductive assumption. This proves that up to a scalar multiple we have produced a splitting, and by construction it has weight 0 so that it must be canonical because of the position of the weights of $\mathrm{End}_F(Z_N, D_N)$. (See proof of 4.3.17.) $\square$

Proposition A.4.8 (A.1.9 part 2.) *Let P and Q be parabolic subgroups. There is a canonical splitting on $G \times^B (G/P \times G/Q)$ which is compatible with all double Schubert varieties.*

Proof: Choose a reduced expression of a minimal representative of w_0 modulo the Weyl group of P. Let it be followed by a reduced expression for w_0 and let that finally be followed by a reduced expression for a minimal representative of w_0 modulo the Weyl group of Q. Together that is a long expression based on which one gets a Z_n which maps birationally onto $G \times^P G \times^B G/Q$ by “multiplication”. This proper birational map has the direct image property because the target is normal. One now takes the canonical splitting of A.4.7. It is compatible with all unions of intersections of components of D_n.

Next note that $G \times^B (G/P \times G/Q) \overset{\phi}{\approx} G \times^P G \times^B G/Q$. The map ϕ is defined by $\phi(g, \overline{x}, \overline{y}) = (gx, x^{-1}, \overline{y})$ (cf. 1.2.2). The image of $G \times^B (X_v \times X_w)$ under ϕ is $G \times^P (\overline{Pv^{-1}B}) \times^B X_w$, which is clearly the image of an intersection of components of D_n. So the splitting is compatible with it. $\square$

We are also ready to prove

Proposition A.4.9 (A.1.9 part 1.) *The product $G/B \times G/B$ is Frobenius split. Further the diagonal $\Delta = \{(x,x)|\, x \in G/B\}$ is compatibly split in $G/B \times G/B$.*

Proof: Take $Q = G$, $P = B$ in the previous proof and recall (1.2.2) that $G \times^B G/B \overset{\phi}{\approx} G/B \times G/B$ with $\phi(g, \overline{h}) = (\overline{g}, \overline{gh})$. We get a splitting which is compatible with $G \times^B B/B$, and that subspace is mapped to the diagonal by ϕ. $\square$

A.5 Two Technical Results

Sublemma A.5.1 *Let X, Y be two quasi-projective schemes over an algebraically closed field k of characteristic $p > 0$. Let $f : X \to Y$ be a bijective proper morphism. Then for every line bundle $\mathcal{L}$ on Y and for $s \in H^0(X, f^*(\mathcal{L}))$ we have $s^{p^n} \in \mathrm{image}(H^0(Y, \mathcal{L}^{p^n}))$ for some large n.*

Proof: As f is proper and quasi-finite, it is finite and affine. We may assume X and Y to be reduced, in which case $H^0(Y, \mathcal{L}^n)$ may be identified with its image. Then the problem is local on Y. Thus we may assume that Y and X are affine and that the line bundles are trivial. We identify them with the structure sheafs. Say $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $A \subset B$. [See erratum.99] As B is finite over A, we have a bound on the dimension of $B \otimes_\phi k$ for any point $\phi : A \to k$. We may replace B by B^pA. Repeating that if necessary we may assume that for all points ϕ the local artin algebra $B \otimes_\phi k$ is reduced. But then it must simply be k, as k is algebraically closed. By Nakayama's Lemma the map $A \to B$ is now surjective at all points, hence surjective. $\square$

Proposition A.5.2 *Let $f : X \to Y$ be a surjective, separable, proper morphism between two varieties, with connected fibres. We assume that Y is Frobenius split. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Proof: By Stein factorisation we may assume f to be finite. Then it is actually a bijection, so that our earlier Lemma A.5.1 applies. We may assume again that $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$, $A \subset B$ and we have to show that A is p-root closed in B.

First consider a smooth point x of X, such that the tangent map is surjective at x and $f(x)$ is smooth in Y. As the dimensions are the same at x and $f(x)$, the surjectivity of the tangent map implies an "analytic" isomorphism $\widehat{\mathcal{O}}_x \approx \widehat{\mathcal{O}}_{f(x)}$. Thus $f(x)$ is outside the support of the A-module B/A. Therefore there is $c \in A$ which annihilates that module—one says c is in the conductor of B over A—such that $c(f(x)) \neq 0$.

Return to the question of p-root closure. Let $b \in B$ with $b^p \in A$ and let $\sigma : A \to A$ be the splitting. For c in the conductor we have cb, c, $b^p \in A$, so $c\sigma(b^p) = \sigma(c^p b^p) = cb$. So b equals $\sigma(b^p)$ at all points where c does not vanish. Varying c we get a dense set of points where b equals $\sigma(b^p)$, so $b \in \sigma(A) \subset A$. $\square$

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Glossary of Notations

$(\ ,\ )$	W-invariant inner product, 4
$w' \leq w$	w' precedes w in the Bruhat order, *i.e.* $X_{w'} \subset X_w$, 17
$s\mu > \mu$	$(\rho - s\rho, s\mu - \mu) > 0$, 71
μ_1	anti-dominant weight in W-orbit of μ, 18
ν_0	dominant weight in W-orbit of ν, 17
$\mathrm{ind}_P^G \circ \mathrm{ind}_B^P$	composite functor, 11
$a * \sigma$	$b \mapsto \sigma(a \cdot b)$ when a is ring element, 39
$g * \sigma$	$b \mapsto g \cdot \sigma(g^{-1} \cdot a)$ when g is group element, 39
$m^*\mathcal{L}$	pullback of $\mathcal{L}$, see [7], 14
$m_*\mathcal{O}$	direct image of $\mathcal{O}$, 14
M^*	dual of M, 25
$(F_*\mathcal{O}_X)^*$	$\mathcal{H}om(F_*\mathcal{O}_X, \mathcal{O}_X)$, 42
$G \times^B X$	total space of associated fibre bundle, 7
$\mathcal{L}\|_X$	restriction of bundle to subspace X, 11
$A_{<\lambda}$	$\bigoplus_i A^i_{<i\lambda}$ when A is graded, 41
$A_{\leq\lambda}$	also in graded case: $\oplus_i A^i_{\leq i\lambda}$, 35
$M_{<\lambda}$	largest B-submodule of M that is in $\mathcal{C}_{<\lambda}$, 23
$M_{\leq R}$	largest B-submodule with weights of length $\leq R$, 10
$M_{\leq\lambda}$	largest B-submodule of M that is in $\mathcal{C}_{\leq\lambda}$, 23
M_μ	weight space of weight μ, 2
$G_{\mathbb{Z}}$	$\mathbb{Z}$-form of G, 68
B	Borel subgroup, 2
BwB	double coset, 4
$\mathcal{C}_B$	the category of rational B-modules, 10
$\mathcal{C}_G$	the category of rational G-modules, 9
$\mathcal{C}_{\leq\lambda}$	category of B-modules whose weights precede λ, 23
$\mathcal{C}_{<\lambda}$	subcategory with weights strictly preceding λ, 22
$\mathcal{C}_{\leq R}$	subcategory of $\mathcal{C}_B$ with length of weights $\leq R$, 10

$\mathcal{L}(\lambda)$	$\mathcal{L}(k_\lambda)$, 15
$\mathcal{L}(\lambda, \mu, \nu)$	line bundle $\mathcal{L}(\lambda) \times \mathcal{L}(\mu) \times \mathcal{L}(\nu)$, 49
n	has no fixed value, 44
$\mathcal{O}(n)$	power of twisting sheaf [7], 27
$\mathcal{O}_X$	structure sheaf of X, 14
p	the characteristic, 33
$\mathbb{P}^n$	projective n-space [7], 26
$P(\mu)$	dual Joseph module with socle k_μ, 18
P_i	minimal parabolic $Bs_iB \cup B$, 4
P_s	minimal parabolic $BsB \cup B$, 5
P_μ	parabolic with $\mathcal{L}(\mu)$ very ample on G/P_μ, 49
$Q(\mu)$	minimal relative Schubert module with socle k_μ, 19
$Q(S, S', \lambda)$	$\ker(\mathrm{res} : H^0(S, \mathcal{L}(\lambda)) \to H^0(S', \mathcal{L}(\lambda)))$, 19
R^nF	n-th derived functor of F, 26
$R_u(G)$	unipotent radical of G, 2
res^G_H	restriction functor $\mathcal{C}_G \to \mathcal{C}_H$, 10
$\mathrm{SL}(n, k)$	special linear group, 1
$\mathrm{soc}\, M$	socle of M, usually as B-module, 10
s_i	i-th simple reflection in a sequence, 3
T	maximal torus, contained in B, 4
U	unipotent radical of B, 5
U_α	root subgroup $\{x_\alpha(t) \mid t \in k\}$, 40
W	Weyl group, 3
w_0	longest element, 7
$X(G)$	character group of G, 2
$X(T)^+$	the set of dominant weights in $X(T)$, 17
$X(T)^-$	the set of anti-dominant weights in $X(T)$, 9
X_w	the Schubert variety $\overline{BwB}/B$, 6
∂X_w	complement of the open Bruhat cell in X_w, 19
x_β	isomorphism $\mathbb{G}_a \to U_\beta$, 4
Z_j	Demazure resolution, 6
ρ	half sum of the roots of B, 27
Σ_1	union of double Schubert varieties, 49
ω_X	Ω^n_X where X is smooth of dimension n, 44
Ω^q_X	sheaf of q-forms on X, 44

Index

First some notions for which we refer to textbooks, as indicated.

Now the terms that are explained in the notes.

September 29, 2005

see also pages 181-182 in
Documenta Mathematica, Extra Volume Suslin (2010).

Frank Grosshans has pointed out that the proof of sublemma A.5.1 is not convincing after the reduction to the affine case.

Let me take another way, much more slowly, making sure that this time there is an actual proof. If I remember correctly the argument below is basically the original one. Sometimes it is better not to simplify.

So we are at the stage where $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $A \subset B$. Both A and B are of finite type over the algebraically closed field k of characteristic $p > 0$, B is finite over A, $X \to Y$ is bijective (between k valued points). [We will not use that it is actually a bijection of scheme theoretic points.] Then sublemma A.5.1 claims that for all $b \in B$ there is an m with $b^{p^m} \in A$. We will argue by induction on the Krull dimension of A.

Say B as an A-module is generated by d elements $b_1, \ldots, b_d$. Let $\mathfrak{p}_1, \ldots \mathfrak{p}_s$ be the minimal prime ideals of A.

Suppose we can show that for every i, j we have $m_{i,j}$ so that $b_j^{p^{m_{i,j}}} \in A+\mathfrak{p}_iB$. Then for every i we have m_i so that $b^{p^{m_i}} \in A+\mathfrak{p}_iB$ for every $b \in B$. Then $b^{p^{m_1+\cdots m_s}} \in A+\mathfrak{p}_1\cdots\mathfrak{p}_sB$ for every $b \in B$. As $\mathfrak{p}_1\cdots\mathfrak{p}_s$ is nilpotent, one finds m with $b^{p^m} \in A$ for all $b \in B$. The upshot is that it suffices to prove the sublemma for the inclusion $A/\mathfrak{p}_i \subset B/\mathfrak{p}_iB$. [It is an inclusion because there is a prime ideal $\mathfrak{q}_i$ in B with $A \cap \mathfrak{q}_i = \mathfrak{p}_i$.] Therefore we further assume that A is a domain.

Let $\mathfrak{r}$ denote the nilradical of B. If we can show that for all $b \in B$ there is m with $b^{p^m} \in A + \mathfrak{r}$, then clearly we can also find an u with $b^{p^u} \in A$. So we may as well replace $A \subset B$ with $A \subset B/\mathfrak{r}$ and assume that B is reduced. But then at least one component of $\mathrm{Spec}(B)$ must map onto $\mathrm{Spec}(A)$, so bijectivity implies there is only one component. In other words, B is also a domain.

Choose t so that the field extension $\mathrm{Frac}(A) \subset \mathrm{Frac}(AB^{p^t})$ is separable. (So it is the separable closure of $\mathrm{Frac}(A)$ in $\mathrm{Frac}(B)$.) As $X \to \mathrm{Spec}(AB^{p^t})$ is also bijective, we have that $\mathrm{Spec}(AB^{p^t}) \to \mathrm{Spec}(A)$ is bijective. It clearly suffices to prove the sublemma for $A \subset AB^{p^t}$. So we replace B with AB^{p^t} and further assume that $\mathrm{Frac}(B)$ is separable over $\mathrm{Frac}(A)$.

Now the idea is that $X \to Y$ has a degree which is the degree of the separable field extension. But the degree must be one because of bijectivity.

Suppose, to contradict, $\mathrm{Frac}(B) \neq \mathrm{Frac}(A)$. Choose b in B outside $\mathrm{Frac}(A)$. It has a separable minimal polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over $\mathrm{Frac}(A)$, with $a_i \in A$. We localize to make it monic and then invert its discriminant: As f is separable, there is a nonzero s' in the intersection of A with $f(x)A[x] + f'(x)A[x]$. Put $s = a_n s'$. The maps $\mathrm{Spec}(B[1/s]) \to \mathrm{Spec}(A[b][1/s])$ and $\mathrm{Spec}(A[b][1/s]) \to \mathrm{Spec}(A[1/s])$ are surjective and their composite is bijective, so both are bijective. The ring $A[b][1/s]$ is a free $A[1/s]$-module with basis 1, b, $\ldots b^{n-1}$. Choose $\phi : A[1/s] \to k$. We have that $A[b][1/s] \otimes_\phi k$ equals $k[x]/(\phi(a_n)x^n + \phi(a_{n-1})x^{n-1} + \cdots + \phi(a_0))$, which has more than one maximal ideal because the polynomial $\phi(a_n)x^n + \phi(a_{n-1})x^{n-1} + \cdots + \phi(a_0)$ is separable and k is algebraically closed. We have arrived at the desired contradiction.

So we now are considering the case that $\mathrm{Frac}(B) = \mathrm{Frac}(A)$. Let Let $\mathfrak{c}$ be the conductor of $A \subset B$. So $\mathfrak{c} = \{\ b \in B \mid bB \subset A\ \}$. We know it is nonzero. If it is the unit ideal then we are done. Suppose it is not. By induction applied to $A/\mathfrak{c} \subset B/\mathfrak{c}$ (we need the induction hypothesis for the original problem without any of the intermediate simplifications) we have that for each $b \in B$ there is an m so that $b^{p^m} \in A + \mathfrak{c} = A$. We are done.

WvdK