MATHEMATICS

# ANOTHER PRESENTATION FOR STEINBERG GROUPS

BY

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## § 1. INTRODUCTION

Let $R$ be a commutative ring with identity, $n \geqslant 3$. Suslin has shown that the elementary subgroup $E(n, R)$ is normal in the general linear group $GL(n, R)$. In other words, $E(n, R)$ is invariant under change of co-ordinates. Here we will establish the analogue for the Steinberg group $St(n, R)$, when $n \geqslant 4$. We will give a presentation for $St(n, R)$ which is invariant under change of co-ordinates. Thus a change of co-ordinates, given by an element $M$ of $GL(n, R)$, will induce an automorphism $\alpha_M$ of $St(n, R)$. This $\alpha_M$ is compatible with inner conjugation by $M$ in $GL(n, R)$. If $M$ is the image of some element $x$ of $St(n, R)$ then $\alpha_M$ is just inner conjugation by $x$. It follows that $K_2(n, R)$ is central in $St(n, R)$, and, if $n \geqslant 5$, that $St(n, R)$ is the universal central extension of $E(n, R)$.

I am indebted to Keith Dennis for suggesting this work and formulating relevant questions when it was in progress.

## § 2. THE RESULTS

2.1. Throughout $R$ is a commutative ring with identity. (For non-commutative rings the proofs fail, especially in 3.2). Let $n \geqslant 4$.

Definitions. Let $U$ be the set of pairs $(i, j)$ with $i$ a unimodular column of length $n$, $j$ a row of length $n$ such that $ji = 0$. For $(i, j) \in U$ we put $e(i, j) = 1 + ij$, where 1 is the identity matrix in $GL(n, R)$. So $e(i, j)v = v + i(jv)$, if $v$ is a column of length $n$. (Note that $jv \in R$). And also $w\, e(i, j) = w + (wi)j$, if $w$ is a row of length $n$. We have $(ij)^2 = 0$, so $e(i, j) \in GL(n, R)$.

2.2. Definition. $St^*(n, R)$ is the group defined by the following presentation.

Generators: $X(i, j)$ with $(i, j) \in U$.

Relations:

$$X(i, j)X(i, k) = X(i, j+k) \text{ if } (i, j), (i, k) \in U.$$
$$X(i, j)X(k, l)X(i, j)^{-1} = X(k + i(jk), l - (li)j), \text{ if } (i, j), (k, l) \in U.$$

Note that $X(k + i(jk), l - (li)j) = X(e(i, j)k, l\, e(i, j)^{-1})$.

2.3. Remark. One may want to generalize the definition to the case where $R^n$ is replaced by a finitely generated projective $R$-module $P$, with dual $P^*$. For $U$ one then takes the set of pairs $(i, j)$ with $i$ unimodular in $P$, $j \in P^*$ such that $ji = 0$. (Recall that $i$ is called unimodular if there is $k \in P^*$ with $ki = 1$).

2.4. Notations. Let $\varepsilon_p$ denote the $p$-th basis vector of $R^n$, i.e. the column with 1 at place $p$ and zeroes elsewhere. Let $\varepsilon_p^T$ denote the transpose of $\varepsilon_p$. The usual generators $e_{pq}(a)$ of $E(n, R)$ can now also be written as $e(\varepsilon_p, a\varepsilon_q^T)$. Let $\pi\colon St^*(n, R) \to GL(n, R)$ denote the natural homomorphism which sends $X(i, j)$ to $e(i, j)$. We also denote by $\pi$ the natural homomorphism $St(n, R) \to GL(n, R)$ which sends $x_{pq}(a)$ to $e_{pq}(a)$.

2.5. Theorem 1. *Let $n \geqslant 4$. There is an isomorphism $St(n, R) \to$ $\to St^*(n, R)$, sending $x_{pq}(a)$ to $X(\varepsilon_p, a\varepsilon_q^T)$.*

2.6. Corollary 1. If $n \geqslant 4$, $K_2(n, R)$ is central in $St(n, R)$.

Proof. It is easy to see that $xX(k, l)x^{-1} = X(\pi(x)k, l\pi(x)^{-1})$ for $x \in St^*(n, R)$, $(k, l) \in U$. Therefore $\ker \pi$ is central in $St^*(n, R)$. Now apply the theorem.

2.7. Corollary 2. If $n \geqslant 5$, $St(n, R)$ is a universal central extension of $E(n, R)$.

Proof. See [4], remark to theorem 5.10.

2.8. Corollary 3. If $n = 4$ and $R$ has no residue field with two elements, $St(n, R)$ is a universal central extension of $E(n, R)$.

Proof. See [3] Theorem (2.6) and [4] Theorem 5.3.

2.9. Corollary 4. Let $M \in GL(n, R)$ and let $\beta_M$ denote inner conjugation by $M$ in $GL(n, R)$. There is one and only one homomorphism $\alpha_M\colon St(n, R) \to St(n, R)$ that makes the following diagram commute:

$$\begin{array}{ccc} St(n, R) & \xrightarrow{\alpha_M} & St(n, R) \\ \downarrow \pi & & \downarrow \pi \\ GL(n, R) & \xrightarrow[\beta_M]{} & GL(n, R) \end{array}$$

Remarks. If $n \geqslant 5$ Corollary 4 follows from Corollary 2 and the fact that $E(n, R)$ is normal in $GL(n, R)$.

Conversely, it follows from Corollary 4 that $E(n, R)$ is normal in $GL(n, R)$, but Suslin's proof of the latter is included in the proof of Theorem 1.

PROOF OF COROLLARY 4. Uniqueness follows from Corollary 1 and the fact that $St(n, R)$ is perfect. (See [4], Lemma 5.4). To prove existence one factors over $St^*(n, R)$, where one sends $X(i, j)$ to $X(Mi, jM^{-1})$.

2.10. Recall that $St(n, R)$ admits an automorphism called "transpose inverse", which sends $x_{pq}(a)$ to $x_{qp}(-a)$. This automorphism has no convenient description in $St^*(n, R)$ because $U$ is not closed under the operation $(i, j) \to (j^T, i^T)$. (Recall that $^T$ stands for "transpose"). But $U$ is not the only set of pairs $(i, j)$ for which one can prove results like Theorem 1. We give an example.

2.11. DEFINITION. Let $V$ be the set of pairs $(i, j)$ with
(a) $i$ is a column of length $n$.
(b) $j$ is a row of length $n$.
(c) $ji = 0$.
(d) There is $M \in GL(n, R)$ such that both $Mi$ and $jM^{-1}$ have at least two zeroes.

Let $St^\wedge(n, R)$ be the group defined by the following presentation.
Generators: $Y(i, j)$ with $(i, j) \in V$.
Relations:

$$Y(i, j)Y(i, k) = Y(i, j+k) \text{ if } (i, j), (i, k), (i, j+k) \in V.$$
$$Y(i, k)Y(j, k) = Y(i+j, k) \text{ if } (i, k), (j, k), (i+j, k) \in V.$$
$$Y(i, j)Y(k, l)Y(i, j)^{-1} = Y(k+i(jk), l-(li)j) \text{ if } (i, j), (k, l) \in V.$$

2.12. THEOREM 2. *Let $n \geqslant 4$. There is an isomorphism $St(n, R) \to$ $\to St^\wedge(n, R)$, sending $x_{pq}(a)$ to $Y(\varepsilon_p, a\varepsilon_q^T)$.*

2.13. In $St^\wedge(n, R)$ the "transpose inverse" automorphism can be described by $Y(i, j) \mapsto Y(j^T, i^T)^{-1}$.

2.14. We leave it to the reader to select his own favorite set of pairs $(i, j)$ and see what goes through for that set.

2.15. REMARK 1. It is not always true that $\alpha_M x = x$ for $x \in K_2(n, R)$. (This would be the case if $K_2(n, R) \to K_2(R)$ were injective and also if we had $M \in E(n, R)$). Counter examples can be obtained from [2], 7.18–7.21, using tables of homotopy groups of spheres, with $M$ a diagonal matrix whose diagonal is $(-1, 1, 1, \ldots, 1)$.

REMARK 2. Even if $R$ is not commutative there is an action of $GL(n, R)$ on $St(n+2, R)$, for $n \geqslant 1$. This can be seen by means of a variation on Theorem B′ of [1]. Instead of using the $x_{ij}(r)$ with $|i-j| \leqslant 2$ as generators, one now uses the $x_{ij}(r)$ with $i > n$ or $j > n$ (and, as always, $i \neq j$). Relations are those Steinberg relations which involve only the chosen generators.

With this presentation it is not hard to define an action of $GL(n, R)$. (cf. proof of Corollary 4).

## § 3. PROOF OF THE THEOREMS

3.1. We write $i \in R^n$ to indicate that $i$ is a column of length $n$ and we write $j^T \in R^n$ to indicate that $j$ is a row of length $n$. Say $i, j^T, k^T \in R^n$. Put $w_{pq} = (j_p k_q - j_q k_p)(i_q \varepsilon_p^T - i_p \varepsilon_q^T)$. Here $i_q$, $k_p$ are co-ordinates of $i$, $k$ resp., and the result is a row with at least $n-2$ zeroes. Note that $w_{pq} = w_{qp}$, $w_{pp} = 0$.

3.2. LEMMA. $w_{pq} i = 0$ and $\sum_{p<q} w_{pq} = (ki)j - (ji)k$.

PROOF. Straightforward.

3.3. Now assume $(i, j) \in U$ and choose $k$ such that $ki = 1$. Then we find $\sum_{p<q} w_{pq} = j$ and the $(i, w_{pq})$ are in $U$. So $X(i, j)$ is the product of the $X(i, w_{pq})$. As $n \geqslant 4$, the $w_{pq}$ have at least two zeroes. (In Suslin's proof that $E(n, R)$ is normal in $GL(n, R)$ one only needs one zero. Therefore he only requires $n \geqslant 3$).

3.4. LEMMA. $St^*(n, R)$ is perfect.

PROOF. By 3.3 it is sufficient to show that $X(i, w)$ is in the commutator subgroup when $(i, w) \in U$ and $w$ has at least two zeroes. Say $w_1 = w_2 = 0$. Suppose $j, v$ are such that $(i, j), (v, w) \in U$. Then

$$[X(i, j), X(v, w)] = X(i, j)X(i, j - (jv)w)^{-1} = X(i, (jv)w).$$

In particular, if $ji = 0$ and $v = \varepsilon_1$, then $X(i, j_1 w)$ is a commutator. Similarly $X(i, j_2 w)$ is a commutator. So we will be done if the ideal $J$ generated by the possible values of $j_1$ and $j_2$ is the unit ideal. Taking $j = i_p \varepsilon_q^T - i_q \varepsilon_p^T$ one sees that $J$ contains the co-ordinates of $i$. Now recall that $i$ is unimodular.

3.5. It is easy to see that $x_{pq}(a) \mapsto X(\varepsilon_p, a\varepsilon_q^T)$ defines a homomorphism $\phi: St(n, R) \to St^*(n, R)$.

LEMMA. To prove Theorem 1 it is sufficient to find a homomorphism $\psi: St^*(n, R) \to St(n, R)$ so that $\psi\phi$ is the identity and so that $\pi\psi = \pi$.

PROOF. Assume we have $\psi$. Then $\pi\phi\psi = \pi$. But $\pi: St^*(n, R) \to E(n, R)$ is a central extension (see proof of Corollary 1) and $St^*(n, R)$ is perfect, so $\phi\psi$ is the identity, by [4] lemma 5.4. The theorem follows.

3.6. In order to obtain $\psi$ it is sufficient to find elements $X(i, j)$ in $St(n, R)$ such that

(a) $X(i,j)$ is defined when $(i,j)\in U$, and $\pi(X(i,j))=e(i,j)$.
(b) $X(i,j)X(i,k)=X(i,j+k)$ if $(i,j),(i,k)\in U$.
(c) $X(i,j)X(k,l)X(i,j)^{-1}=X(k+i(jk), l-(li)j)$ if $(i,j),(k,l)\in U$.
(d) $X(\varepsilon_p, a\varepsilon_q^T)=x_{pq}(a)$.

Note that we used the notation $X(i,j)$ before to denote the generators of $St^*(n,R)$. There will be no confusion as we will not need $St^*(n,R)$ any more; the computations and definitions in the sequel all refer to $St(n,R)$.

3.7. Notation. If $i\in R^n$, $1\leqslant r\leqslant n$, set $x(i)_r=\prod_{p\neq r}x_{pr}(i_p)$. So $x(i)_r$ is a product of factors with "column index" $r$. We can ignore the co-ordinate $i_r$. One may also replace $i$ by $i'$ where $i'$ has a zero at place $r$ and the same co-ordinates as $i$ otherwise. Clearly $x(i)_r=x(i')_r$.

If $j^T\in R^n$, $1\leqslant r\leqslant n$, set $x_r(j)=\prod_{p\neq r}x_{rp}(j_p)$. Here the "row index" of the factors is $r$. The following well known fact is very useful in computations. Let $j^T\in R^n$, $j_r=0$, and let $z$ be a product of factors with column index different from $r$. Then $zx_r(j)z^{-1}=x_r(j\pi(z)^{-1})$. (If the factors do not have row index $r$ either, the condition $j_r=0$ is superfluous). Similarly $yx(i)_ry^{-1}=x(\pi(y)i)_r$ if $i_r=0$ and $y$ can be written as a product of factors with row index different from $r$. (We will meet situations where an element can be written two ways. It is of course sufficient if one of these two ways satisfies the criterion). Also note the rules $x(i+k)_r=x(i)_rx(k)_r$ and $x_r(j+l)=x_r(j)x_r(l)$.

3.8. Definition. Let $i,j^T\in R^n$, $ji=0$, $1\leqslant r\leqslant n$, $i_r=0$. Then set $x(i,j)=[x(i)_r, x_r(j)]x(ij_r)_r$. It is easy to see that $\pi(x(i,j))=e(i,j)$. We have to show that the definition is consistent, i.e. that if $i_s$ is also zero, $[x(i)_r, x_r(j)]x(ij_r)_r=[x(i)_s, x_s(j)]x(ij_s)_s$.

3.9. Say $r=1$, $s=2$. Write $j$ as $a\varepsilon_1^T+b\varepsilon_2^T+k$, where $k_1=k_2=0$. Put $y=[x(i)_1, x_1(k)]$. Then $y$ is a product of factors with row index different from 2, as $i_2=0$. Therefore $yx_{12}(b)y^{-1}=x(\pi(y)b\varepsilon_1)_2=x_{12}(b)$. Similarly

$$yx(ib)_2y^{-1}=x(ib)_2,\qquad [x(i)_1, x_{12}(b)]=x(ib)_2,\qquad [x(ia)_1, x(ib)_2]=1.$$

So

$$[x(i)_1, x_1(j)]x(ia)_1=[x(i)_1, x_{12}(b)]x_{12}(b)[x(i)_1, x_1(k)]x_{12}(b)^{-1}x(ia)_1=$$
$$=x(ib)_2yx(ia)_1=yx(ib)_2x(ia)_1=yx(ia)_1x(ib)_2.$$

Interchanging the roles of 1 and 2 yields

$$[x(i)_2, x_2(j)]x(ib)_2=[x(i)_2, x_2(k)]x(ia)_1x(ib)_2.$$

So it remains to show that $y=[x(i)_2, x_2(k)]$. Just as $y$ commutes with $x_{12}(b)$ it commutes with $x_{12}(1)$. It also commutes with $x_{21}(1)$. (Apply "transpose inverse" or use that $y$ is a product of factors with column index different from 2). So $y$ commutes with $w_{12}(1)=x_{12}(1)x_{21}(1)^{-1}x_{12}(1)$, and $y=w_{12}(1)yw_{12}(1)^{-1}=[x(i)_2, x_2(k)]$, as required.

3.10. DEFINITION. Let $i, j^T \in R^n$, $ji=0$, $1 \leqslant r \leqslant n$, $j_r=0$. Then set $x(i,j)=x_r(i_r j)[x(i)_r, x_r(j)]$. Again $\pi(x(i,j))=e(i,j)$. The definition is internally consistent for reasons similar to those given above. One can also use that the "transpose inverse" automorphism sends the present $x(i,j)$ to the inverse of $x(-j^T, -i^T)$, where the latter is taken in the sense of 3.8. Remains to show that definition 3.10 is consistent with definition 3.8 when both apply. If $i_r=j_r=0$ this is obvious. So we are already free to use both definitions of $x(v, w)$ when $v_r=w_r=0$ for some $r$. Now say $i_1=j_2=0$. Write $j=a\varepsilon_1^T+k$, $i=c\varepsilon_2+l$, where $k_1=k_2=l_1=l_2=0$. We have to show that $[x(i)_1, x_1(j)]x(ia)_1=x_2(cj)[x(i)_2, x_2(j)]$, or that $x(i, k)x(ia)_1= =x_2(cj)x(l,j)$. The left hand side equals

$$x_2(ck)[x(i)_2, x_2(k)]x(ia)_1=x_2(ck)x(l, k)x_{21}(ca)x(la)_1,$$

the right hand side equals

$$x_2(cj)[x(l)_1, x_1(j)]x(la)_1=x_2(ck)x_{21}(ca)x(l, k)x(la)_1.$$

So we need that $x(l, k)$ commutes with $x_{21}(ca)$. It does, by the usual argument. The trick in these computations is to apply the definitions 3.8, 3.10 with different values of $r$, in order to rewrite commutators. Thus one can break some commutators into pieces. Other commutators can be rewritten so that a certain row or column index is avoided. In the sequel these manipulations will be left to the reader.

3.11. LEMMA. Let $i, j^T, k^T \in R^n$, $ji=ki=0$. Assume either that $i$ has at least two zeroes, or that there are $p, q, r$, distinct, with $j_r=j_p=k_p=k_q=0$. Then $x(i,j)x(i, k)=x(i, j+k)$. A similar statement holds with rows and columns interchanged. (e.g. apply "transpose inverse").

PROOF. First let $i_p=i_q=0$, $p \neq q$. Then

$$x(i, j+k)=[x(i)_p, x_p(j)]x_p(j)[x(i)_p, x_p(k)]x_p(j)^{-1}x(ij_p+ik_p)_p.$$

Decomposing the second commutator one sees that it can be written without row index $p$ and also without column index $p$. The result follows easily. Next let $j_r=j_p=k_p=k_q=0$, $p$, $q$, $r$ distinct. Again the commutators $[x(i)_p, x_p(j)]$, $[x(i)_p, x_p(k)]$ can be written without column index $p$ and the result follows easily.

3.12. LEMMA. Let $i, j^T \in R^n$, $y=x_{pq}(a)$. (So $y$ is one of the ordinary generators of $St(n, R)$). If $j$ has at least two zeroes and $ji=0$, then $yx(i,j)y^{-1}=x(\pi(y)i, j\pi(y)^{-1})$.

PROOF. Say $p=1$, $q=2$. One has essentially two cases: $j_3=0$; $j_1=j_2=0$. In each case decompose $x(i,j)$, then conjugate by $y$, then put things together again, using arguments as above.

3.13. Definition. Let $(i, j) \in U$. (see 2.1). We define $\bar{X}(i, j)$ to be the set of $x \in St(n, R)$ that can be written as $\prod_m x(i, w^m)$, where $\sum_m w^m = j$, each $w^m$ is a scalar multiple of one of the rows $i_q \varepsilon_p^T - i_p \varepsilon_q^T$. One may use the same pair $p, q$ repeatedly and one may choose the order in the product. Thus it is obvious that $x \in \bar{X}(i, j)$, $y \in \bar{X}(i, k)$ implies $xy \in \bar{X}(i, j+k)$, when $(i, j)$, $(i, k) \in U$. Our purpose is to show that each set $\bar{X}(i, j)$ consists of exactly one element, which will then be written as $X(i, j)$. The $X(i, j)$ will satisfy the requirements listed in 3.6.

3.14. Lemma. Let $(i, j) \in U$, $y \in St(n, R)$. Then

$$y\bar{X}(i, j)y^{-1} \subseteq \bar{X}(\pi(y)i, j\pi(y)^{-1}).$$

Proof. We may assume $y = x_{pq}(a)$ and it suffices to show that

$$yx(i, w)y^{-1} \in \bar{X}(\pi(y)i, w\pi(y)^{-1})$$

for $w = b(i_r \varepsilon_s^T - i_s \varepsilon_r^T)$. There are a few cases, such as the case $p = r$, $q \neq s$. In each case apply 3.12 and, where necessary, 3.11.

3.15. Lemma. Let $(\varepsilon_1, j) \in U$, $M \in E(n, R)$. Then $\bar{X}(M\varepsilon_1, jM^{-1})$ consists of exactly one element.

Proof. Choose $y \in St(n, R)$ with $\pi(y) = M$. Then

$$y\bar{X}(\varepsilon_1, j)y^{-1} \subseteq \bar{X}(M\varepsilon_1, jM^{-1})$$

and

$$y^{-1}\bar{X}(M\varepsilon_1, jM^{-1})y \subseteq \bar{X}(\varepsilon_1, j),$$

so we may assume $M = 1$. In that case $x(\varepsilon_1, w^m) = x_1(w^m)$ (use $r = 1$ in 3.10) and thus $\prod_m x(\varepsilon_1, w^m) = x_1(j)$.

3.16. Remark. At this stage one can already show that $K_2(n, R)$ is central in $St(n, R)$, by an argument as in 2.6.

3.17. Definition. Let $i, j^T, k^T \in R^n$ with $ji = 0$, $ki = 1$. We would like to define $x_i(j; k)$ as the product of the $x(i, w_{pq})$ with $p < q$, where $w_{pq}$ is defined as in 3.1. The product might depend on the order of the factors however (we will see later that it does not). Therefore we define instead $\bar{x}_i(j; k)$ to be the set of values that one gets when varying the order. From 3.2 it follows that $\bar{x}_i(j; k) \subseteq \bar{X}(i, j)$.

3.18. Lemma. Let $(i, j) \in U$, $k^T \in R^n$ with $ki = 1$. If $j$ has at least two zeroes then $x(i, j) \in \bar{x}_i(j; k) \subseteq \bar{X}(i, j)$.

Proof. · Say $j_1 = j_2 = 0$. The product of the $x(i, w_{1q})$ is $x(i, v)$ where $v = \sum_q w_{1q}$, by 3.11. (Use that the second co-ordinate of $w_{1q}$ is zero). The first co-ordinate of $v$ is zero, by 3.2, so the product of the $x(i, w_{1q})$ is of the form $x(i, l)$ with $l_1 = l_2 = 0$. The same observation holds for the product of the $x(i, w_{2q})$ and also for each of the remaining factors $x(i, w_{pq})$ $(2<p<q)$. So we can take all factors together and obtain $x(i, \sum_{p<q} w_{pq})$, i.e. $x(i,j)$. (The order we used is as follows. First come the $w_{1q}$, then the $w_{2q}$, then the rest.)

3.19. LEMMA. Let $(i,j)$, $(i,k) \in U$ where $j$, $k$ each have at most two non-zero co-ordinates. Then $x(i,j)$, $x(i,k)$ commute.

PROOF. Either 3.11 applies or we are essentially in the following situation: $n=4$, $j_3=j_4=k_1=k_2=0$. Write $i$ as $v^1+v^2+v^3+v^4$ where $v^1=\varepsilon_3$, $v^2=\varepsilon_4$, $v^3=-\varepsilon_3+i_4\varepsilon_4$. Then the $v^r$ are all of the form $M\varepsilon_1$ as in 3.15 and $x(v^r,j) \in \bar{X}(v^r,j)$ by 3.18. We have $x(i,k)x(v^r,j)x(i,k)^{-1} \in \bar{X}(v^r+i(kv^r),j)$, so $x(i,k)x(i,j)x(i,k)^{-1} = \prod_r (x(i,k)x(v^r,j)x(i,k)^{-1}) = \prod_r x(v^r+i(kv^r),j) = x(i,j)$ by 3.11, 3.15, 3.18.

3.20. DEFINITION. By 3.19 there is only one element in $\bar{x}_i(j;k)$. We call it $x_i(j;k)$. Note that $x_i(u;k)x_i(v;k)=x_i(u+v;k)$.

REMARK. For $n>4$ we do not need Lemma 3.15 to prove Lemma 3.19. Then $x_i(j;k)$ can be defined immediately after 3.11. One can then proceed with 3.18, 3.13, 3.21 and only then discuss 3.12, 3.14. In other words, our introduction of the *sets* $\bar{X}(i,j)$, $\bar{x}_i(j;k)$, instead of the *elements* $X(i,j)$, $x_i(j;k)$, is only relevant for $n=4$.

3.21. LEMMA-DEFINITION. Let $(i,j) \in U$. Then $\bar{X}(i,j)$ consists of exactly one element. We call it $X(i,j)$.

PROOF. Choose $k$ such that $ki=1$. Then

$$\prod_m x(i, w^m) = \prod_m x_i(w^m; k) = x_i(\textstyle\sum_m w^m; k) = x_i(j;k)$$

if the $w^m$ are as in 3.13. So $x_i(j;k)$ is the unique element of $\bar{X}(i,j)$.

3.22. It is easy to check that the $X(i,j)$ satisfy the requirements listed in 3.6, so Theorem 1 is proved.

3.23. DEFINITION. Let $i, j^T \in R^n$ with $ji=0$. Assume there is $M \in GL(n,R)$ such that $jM$ has at least two zeroes. Choose columns $v^r$ such that $\sum_r v^r = i$, $(v^r, j) \in U$. (This is possible, cf. proof of Lemma 3.19). We set $Z(i,j) = \prod_r X(v^r,j)$. We need to show that $Z(i,j)$ does not depend on the choice of the $v^r$. We claim that, independent of this choice, $Z(i,j) = \alpha_M(x(M^{-1}i, jM))$, where $\alpha_M$ is as in 2.9. For, by 3.18, $X(M^{-1}v^r, jM) = x(M^{-1}v^r, jM)$, and the product of the $x(M^{-1}v^r, jM)$ is $x(M^{-1}i, jM)$, by 3.11. Also, by construction, $\alpha_M(X(M^{-1}v^r, jM)) = X(v^r,j)$. (See proof of 2.9). The claim follows. From the claim one also sees that $Z(i,j)$ could have been defined as $\alpha_M(x(M^{-1}i, jM))$.

3.24. Definition. Let $(i, j) \in V$. (see 2.11). Define $Y(i, j) = Z(i, j)$. In other words, choose $M$ such that $jM$ has at least two zeroes and put $Y(i, j) = \alpha_M(x(M^{-1}i, jM))$. Note that the generators of $St\^{}(n, R)$ are also called $Y(i, j)$. Clearly $Y(i, k)Y(j, k) = Y(i+j, k)$ when $(i, k)$, $(j, k)$, $(i+j, k) \in V$. Also, $\alpha_M(Y(i, j)) = Y(Mi, jM^{-1})$ when $(i, j) \in V$, $M \in GL(n, R)$. In particular, $Y(i, j)Y(k, l)Y(i, j)^{-1} = Y(k+i(jk), l-(li)j)$ if $(i, j)$, $(k, l) \in V$.

3.25. Let $\tau$ denote the "transpose inverse" involution of $St(n, R)$ (see 2.10) and also the analogous involution of $GL(n, R)$. Let $v$, $w^T \in R^n$, $wv = 0$. If $w$ has at least two zeroes, $\tau(x(v, w)) = x(-w^T, -v^T)^{-1}$. (cf. 3.10). From uniqueness of $\alpha_M$ one sees that $\alpha_M = \tau\ \alpha_{\tau M}\ \tau$, $M \in GL(n, R)$. It follows that $\tau(Y(i, j)) = Y(-j^T, -i^T)^{-1}$ for $(i, j) \in V$. Therefore the $Y(i, j)$ also satisfy the first relation in the list that defines $St\^{}(n, R)$. We get a homomorphism $St\^{}(n, R) \to St(n, R)$, sending $Y(i, j)$ to $Y(i, j)$ for $(i, j) \in V$.

3.26. Lemma. $St\^{}(n, R)$ is perfect.

Proof. If there are $p$, $q$, $r$, distinct, with $i_p = i_r = j_q = j_r = 0$, then $Y(i, j) = [Y(i, \varepsilon_r^T), Y(\varepsilon_r, j)]$. If $(i, j) \in V$ and $i$ has at least two zeroes, we can therefore write $Y(i, j)$ as the product of three commutators. For $M \in GL(n, R)$ there is an automorphism of $St\^{}(n, R)$ sending $Y(i, j)$ to $Y(Mi, jM^{-1})$ for $(i, j) \in V$. The result follows.

3.27. Theorem 2 follows by an argument as in 3.5.

Remark. The homomorphism $St(n, R) \to St\^{}(n, R)$ which sends $x_{pq}(a)$ to $Y(\varepsilon_p, a\varepsilon_q^T)$ can also be described as sending $x_{pq}(a)$ to $Y(a\varepsilon_p, \varepsilon_q^T)$. For, when $r$ is chosen distinct from $p$ and $q$, one has

$$Y(\varepsilon_p, a\varepsilon_q^T) = [Y(\varepsilon_p, \varepsilon_r^T), Y(a\varepsilon_r, \varepsilon_q^T)] = Y(a\varepsilon_p, \varepsilon_q^T).$$

More generally one has $Y(i, aj) = Y(ai, j)$ for $a \in R$, $(i, j) \in V$.

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