STABILITY FOR K_2 OF DEDEKIND RINGS OF ARITHMETIC TYPE

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§1 Introduction

Dunwoody [5] has shown that when R is a euclidean ring the map $K_2(2,R) \to K_2(n,R)$ is surjective for $n \geq 3$. On the other hand Dennis and Stein [4] have given examples where R is a ring of integers in a quadratic imaginary number field and $K_2(2,R) \to K_2(3,R)$ is not surjective. But from the study of K_1 the quadratic imaginary case is known to have particularly bad stability behaviour. (cf. [2], [10], [11]). To be specific, recall that $SL_2(R) = E_2(R)$ when R is a Dedekind ring of arithmetic type with infinitely many units, i.e. when R is the ring of S-integers in a global field, where S is a finite set of places, containing all archimedean places, and $|S| \geq 2$.(See [13]) (Terminology as in Bass-Milnor-Serre [1]). In contrast, there are only five quadratic imaginary number fields whose ring of integers R satisfies $SL_2(R) \doteq E_2(R)$. (The five cases are those with R euclidean.) The main result of this paper is that at the K_2 level the situation is similar:

THEOREM 1. Let R be a Dedekind ring of arithmetic type with infinitely many units. Then $K_2(2,R) \to K_2(R)$ is surjective and $K_2(n,R) \to K_2(R)$ is an isomorphism for $n \geq 3$.

From this theorem and its proof one sees that for the rings in question the group $K_2(R)$ is closely related with $E_2(R)$ and therefore also with such topics as the theory of division chains. (cf. [3], [9]). We hope that a further exploration of this connection will give some useful information on $K_2(R)$. Theorem 1 is proved by expanding our earlier proof of the injective stability theorem for K_2 of finite dimensional noetherian rings. ([6]).(For surjectivity see also §5). This time gereral position arguments are not enough. At a crucial point we need to know, for certain $q,s \in R$, how the "relative elementary subgroup" $E(R,qR)$ of $SL_2(R)$, introduced by Vaserstein in connection with the congruence subgroup problem for SL_2, intersects the congruence subgroup $\ker(SL_2(R) \to SL_2(R/sqR))$. We get the answer from the explicit description (by power norm residue symbols) of the failure of the congruence subgroup property for $SL_2(R)$, as obtained by Vaserstein [13]. (For corrections to the proof see Liehl [8]). So we need some very specific and deep arithmetic information on the ring R in order to get such a sharp bound for the range of stability. We do not need such information in the proof of theorem 2 below (see 2.4). Theorem 2 is a quite general stability theorem for K_2. It is better than the main results in [6] and it is also slightly stronger than the version proved by Suslin and Tulenbayev. (Compare Corollary 2.6). It is no surprise that we recover the Suslin-Tulenbayev Theorem, as we borrow from its proof. But our constructions are described in a different language.

§2 A General Stability Theorem for K_1 and K_2

2.1 Let R be an associative ring with identity. For $n \geq 2$, $q \in R$, we define $U_n(q) = \{a \in R^n$: the column $(1+a_1q, a_2q, \ldots, a_nq)$ is unimodular$\}$.

REMARK. When not stated otherwise, unimodularity will refer to columns, not rows. We should use notations like $(b_1,\ldots,b_n)^T$ for a column, but we simply write $(b_1,\ldots,b_n)$, as in [6]. To get a clear picture the reader has to draw the columns as honest columns anyway.

2.2 We define elementary operations on $U_n(q)$ as follows. (Compare also [1] Ch I §2 and [7] §2).
For $2 \leq i \leq n$, $p \in R$, $a \in U_n(q)$, put
$e_i(p)(a) = (a_1,\ldots,a_{i-1},a_i+p(1+qa_1),a_{i+1},\ldots,a_n)$ and
$e^i(p)(a) = (a_1+pa_i,a_2,\ldots,a_n)$.
These elementary operations $e_i(p)$, $e^i(p)$ generate a group of permutations of $U_n(q)$. The orbit of $(0,\ldots,0)$ under the action of this group is denoted $EU_n(q)$. So an element b of $EU_n(q)$ can be reduced to zero by a finite number of elementary operations. The minimum number that is needed is called the complexity of (b;q), or of b (with respect to q). For instance, $(0,\ldots,0)$ has complexity zero and $(0,\ldots,0,1)$ has complexity one. Several of the constructions below will depend on a choice of the reduction to zero of an element of $EU_n(q)$. We will establish useful properties of these constructions by induction on complexity.

2.3 We say that R satisfies SR_n^2 when the following holds. For any pair of unimodular columns $(a_1,\ldots,a_n)$, $(b_1,\ldots,b_n)$

there are $t_i \in R$ such that both $(a_1+t_1a_n,\dots,a_{n-1}+t_{n-1}a_n)$ and $(b_1+t_1b_n,\dots,b_{n-1}+t_{n-1}b_n)$ are unimodular.

2.4 THEOREM 2.

Let $n \geq 2$ and let R satisfy SR^2_{n+1}. Assume that $EU_n(q)$ equals $U_n(q)$ for all $q \in R$. Then

(i) $K_1(n-1,R) \to K_1(R)$ is surjective and

$K_1(m,R) \to K_1(R)$ is an isomorphism for $m \geq n$.

(ii) $K_2(n,R) \to K_2(R)$ is surjective and

$K_2(m,R) \to K_2(R)$ is an isomorphism for $m \geq n+1$.

2.5 REMARKS. The surjectivity in part (i) is well known. The injectivity of $K_1(m,R) \to K_1(R)$ is also known for $m \geq n+1$. For $m = n$ it can be proved in a traditional fashion, but we will only give an outrageously complicated proof here. Namely, we will get the result as an immediate consequence of our proof of injectivity in part (ii). Surjectivity in part (ii) will also come as a corollary of our proof of injectivity, so all efforts are directed at proving this injectivity for K_2. When one only seeks surjectivity for K_2, there is an easier way, not expounded here. Note that in the situation of Theorem 1 we do describe the easier way. (See section 5).

2.6 We will see in the next section that SR_n (cf. [6] 2.1) implies the hypotheses of Theorem 2. Thus we get

COROLLARY (Suslin, Tulenbayev [12]).

Let R satisfy SR_n, $n \geq 2$. Then $K_2(n,R) \to K_2(R)$ is surjective and $K_2(m,R) \to K_2(R)$ is an isomorphism for $m \geq n+1$.

2.7 Note that in the same fashion part (i) of Theorem 2 implies the standard stability theorem for K_1. From the following application one sees that Theorem 2 may yield a better range than the theorem of Suslin and Tulenbayev. First recall that an integral domain is called totally imaginary if its elements are integral over $\mathbb{Z}$ and its field of fractions is a totally imaginary number field. (cf. [1] and [15] §16).

COROLLARY. Let R be a 1-dimensional commutative ring, finitely generated as a $\mathbb{Z}$-algebra. Assume that for each minimal prime ideal P the domain R/P has infinitely many units and is not totally imaginary. Then $K_2(2,R) \to K_2(3,R)$ is surjective and $K_2(m,R) \to K_2(R)$ is an isomorphism for $m \geq 3$.

PROOF. We will see in the next section that R satisfies SR_3^2. (Prop. 3.8). Remains to show $EU_2(q) = U_2(q)$. This equality is equivalent with the equality $E(R,qR) = G(R,qR)$. (notation of [13]). It is instructive to check this. If R is a Dedekind ring of arithmetic type, the conditions of the corollary make that Vaserstein's theorem applies, so that in fact $E(I,J) = G(I,J)$ for any pair of ideals I, J in R. Therefore, in the general case, one may argue in the fashion of [15] §16, with the E(R,I) of [15] replaced by the E(R,I) of [13]. (cf. [8]).

REMARK. This Corollary implies part of Theorem 1. This part is easier than the remainder of Theorem 1.

§3 More about stable Range Conditions

3.1 In this section we collect some technicalities. For unexplained notation and terminology, see [6].

3.2 When $(a_1,\ldots,a_n) \in U_n(q)$ there are $b_1,\ldots,b_n$ with $b_1(1+a_1q)+b_2a_2q+\ldots+b_na_nq = 1$. Put $c = \sum_1^n b_ia_i$. Then $(1-qc)(1+qa_1) - 1 + \sum_2^n qb_ia_i$ has value zero, so the column $(1+qa_1,a_2,\ldots,a_n)$ is unimodular too.

3.3 LEMMA. If R satisfies SR_n then $EU_n(q)$ equals $U_n(q)$ for all $q \in R$.

PROOF. Fix an orbit in $U_n(q)$. We seek a convenient choice for a representative $(a_1,\ldots,a_n)$ of this orbit. For any choice the column $(1+qa_1,a_2,\ldots,a_n)$ is unimodular, hence we can modify the choice so that $(a_2,\ldots,a_n)$ is unimodular, next so that $a_1 = 0$, finally so that $a = 0$. Therefore it must have been the orbit $EU_n(q)$ of zero.

3.4 LEMMA. SR_n implies SR^2_{n+1} $(n \geq 2)$.

PROOF. Given unimodular columns a,b in R^{n+1} we have to find $v \in R^n$ so that the columns $(a_1+v_1a_{n+1},\ldots,a_n+v_na_{n+1})$, $(b_1+v_1b_{n+1},\ldots,b_n+v_nb_{n+1})$ are unimodular. For any $g \in E([n] \times [n+1])$ we may replace the pair a,b by ga,gb. Using SR_n we may thus reduce to the case $a_n = 1$. By [14] Theorem 1 there is a suitable v with $v_n = 0$, in that case.

3.5 LEMMA. Let $M \in GL_m(R)$, $m \geq 2$. Let $(a_1,\ldots,a_m)$ be the first column of M and $(b_1,\ldots,b_m)$ the last column of M^{-1}. Then $(a_1,\ldots,a_{m-1})$ is unimodular if and only if $(b_2,\ldots,b_n)$ is unimodular.

PROOF. If $(a_1,\ldots,a_{m-1})$ is unimodular, reduce to the case that $a_m = 0$ by multiplying M from the left with a lower triangular matrix. If $(b_2,\ldots,b_m)$ is unimodular, reduce to the case that $b_1 = 0$.

3.6 LEMMA. Let R satisfy SR^2_{n+1}, $n \geq 2$. Then R satisfies SR^3_{n+2} $(n+2,n+2)$.

PROOF. Let $M_1,M_2,M_3 \in GL_{n+2}(R)$. By the previous lemma we will be done if we show that there is $g \in E([n+2] \times \{1\})$ such that in each of the three matrices gM_i^{-1} the part of the last column consisting of the bottom n+1 entries is unimodular. Let a,b,c denote the last column of $M_1^{-1},M_2^{-1},M_3^{-1}$ respectively. We look for $v_i \in R$ such that the column $(a_2+v_2a_1,\ldots,a_{n+2}+v_{n+2}a_1)$ and its two analogues are unimodular. For any $g \in E(\{1\}^* \times [n+2])$ we may replace the triple a,b,c by ga,gb,gc. Using SR_{n+1} we may therefore reduce to the case $a_2 = 1$. Then we want to solve our problem with some v satisfying $v_2 = 0$. We may add multiples of b_2 to $b_3,\ldots,b_{n+2}$ and we may also add multiples of c_2 to $c_3,\ldots,c_{n+2}$. So we may assume $(b_3,\ldots,b_{n+2},b_1)$ and $(c_3,\ldots,c_{n+2},c_1)$ are unimodular. Apply SR^2_{n+1}.

3.7 LEMMA. Let $n \geq 2$. Let R satisfy SR^2_{n+1} and let $GL_n(R)$ act transitively on unimodular columns of length n. Then R satisfies SR^2_{n+1} $(n+2,n+1)$.

PROOF. Let A,B be n by n+1 matrices, each obtained by deleting the bottom row of some element of $GL_{n+1}(R)$. Let $v,w \in R^n$. We want to find an $x \in R^{n+1}$ such that $v+Ax, w+Bx$ are unimodular. Clearly we may replace the system

A,B,v,w by AU,BU,v,w for $U \in GL_{n+1}(R)$ and also by T_1A,T_2B,T_1v,T_2w for $T_i \in GL_n(R)$. From SR^2_{n+1} and Lemma 3.5 it follows that there is $U \in E([n+1] \times \{1\})$ such that the first columns of AU and BU are unimodular. Therefore we may assume A,B have first columns of the form $(1,0,\ldots,0)$ and further that A has first row of the form $(1,0,\ldots,0)$. But then we can choose x so that the first coordinate of $v+Ax$ as well as the second coordinate of $w+Bx$ is equal to one.

3.8 PROPOSITION. Let R be finitely generated as a module over a central subring T whose maximal spectrum is noetherian of dimension $d, d < \infty$. Then R satisfies SR^2_{n+1} for $n \geq \max(2,d+1)$.

PROOF. Recall that R satisfies SR_{n+1} so that $GL_{n+1}(R)$ acts transitively on unimodular columns of length n+1. Using Lemma 3.5 once more we see that SR^2_{n+1} is now equivalent with $SR^2_{n+1}(n+1,n+1)$. By ([6], Theorem 8, pg. 134) it suffices to consider the case that T is a field. But then R satisfies SR_n and Lemma 3.4 applies.

EXERCISE. Give a more direct proof of the Proposition.

§4 Proof of the Theorems

4.1 We start modifying the proof of Theorem 4 of [6] in order to get a proof of Theorems 1 and 2 above. We introduce two sets of hy- theses.

Situation α: $n \geq 2$. The associative ring R satisfies SR^2_{n+1} and $EU_n(q)$ equals $U_n(q)$ for all $q \in R$.

This corresponds with the hypotheses of Theorem 2.

Situation β: $n=2$. The ring R is a Dedekind ring of arithmetic type with infinitely many units.

This corresponds with Theorem 1.

4.2 The proofs for situation α will mostly be simplified versions of those for situation β, but complicated versions of the arguments in [6]. While in [6] the pattern of the proof looks reasonable, the modifications presented here require more perseverance from the reader. From now on we assume that α or β applies. As we are going to use almost all of ([6], sections 2,3,4), we will save some space and refer the reader repeatedly to [6], telling him what to read and when. Of course we now replace the standing as- sumption $\widetilde{\widetilde{SR}}_n$ of [6] by the assumption that α or β holds.

4.3 Read in [6]: All of section 1, 2.1 and 2.2, 3.4 through 3.19. The handwritten $\mathcal{L}$(resp $\mathcal{R}$) of [6] will be denoted by L(resp. R) in this paper.

LEMMA. $E_n(R)$ acts transitively on unimodular clumms of length n.

PROOF. In situation α this follows from the equality of $E\,U_n(1)$ and $U_n(1)$. In situation β it follows from $SL_2(R) = E_2(R)$, which is proved in [13], cf. [8].

4.4 Read [6] 3.20. There is a converse to [6] 3.20:

LEMMA. Let $\langle X,Y\rangle \in C$, $v \in R^{n+1}$. Let $L(x_{n+2}(v))\langle X,Y\rangle$ be defined at the bottom. Choose $T \in St(n+1)$ such that $\langle X,Y\rangle = \langle T\, x_{n+2,1}(*), *\rangle$. Then there is $w \in R^{n+1}$, $r \in R$, with $(w_2,..,w_{n+1})$ unimodular and $x_{n+2}(v)T = Tx_{n+2}(wr)$.

PROOF. Straightforward.

4.5 DEFINITION. When in the situation of lemma 4.4 the column v itself is also unimodular, we say that $L(x_{n+2}(v))\langle X,Y\rangle$ is defined firmly at the bottom. Note that then $(w_2r,..,w_{n+1}r)$ is unimodular so that we may replace w by wr and r by 1. Therefore this is the situation of [6] 3.20.

4.6 In the proof of [6] 3.22 it is used that $K_1(n,R) \to K_1(n+1,R)$ is injective. In situation α this property is not yet established (What is known is that $K_1(m,R) \to K_1(R)$ is injective for $m \geq n+1$. The case m=n is the subject of part (i) of Theorem 2.) Let us therefore give another proof of [6] 3.22. It clearly suffices to show:

LEMMA. Let $v \in R^{n+1}$, $\langle X,Y\rangle \in C$. Let M be the n+1 by 2 matrix whose first column is v and whose second column is obtained from the first column of mat $\langle X,Y\rangle$ by deleting the last entry. Then $L(x_{n+2}(v))\langle X,Y\rangle$ is defined firmly at the bottom if and only if M is completable, by adding columns, to an element of $E(n+1,R)$.

REMARK. Because of lemma 4.3 completability of M is equivalent with unimodularity, i.e. with the existence of a 2 by n+1 matrix N such that NM = id.

PROOF OF LEMMA. Reduce to the case $X = x_{n+2,1}(*)$ and apply lemma 4.3.

4.7 Read [6] 3.23, 3.24, 3.25 and 3.36

LEMMA. Let $q \in R$, $v \in EU_n(q)$. There are $X \in St(\{n+1\}^* \times [n])$, $Y \in St(\{n+1\}^* \times \{1,n+1\}^*)$ such that $x_{1,n+2}(v_1)\dots x_{n,n+2}(v_n)x_{n+2,1}(q) = X\ Y$ in $St(\{n+1\}^* \times \{n+1\}^*)$.

PROOF. By induction on the complexity of $(v;q)$. In this proof let $x(v)$ stand for $x_{1,n+2}(v_1)..x_{n,n+2}(v_n)$ in $St(\{n+1\}^* \times \{n+1\}^*)$.
If $w = e_i(b)v$ has lower complexity than v with respect to q, note that $x(v)\ x_{n+2,1}(q) = x_{i1}(-bq)\ x(w)\ x_{i1}(bq)\ x_{i,n+2}(-b)\ x_{n+2,1}(q) = x_{i1}(-bq)\ x(w)\ x_{n+2,1}(q)\ x_{i,n+2}(-b)$.
If $w = e^i(b)v$ has lower complexity, note that $x(v)\ x_{n+2,1}(q) = x_{1i}(-b)\ x(w)\ x_{n+2,1}(q)\ x_{n+2,i}(-bq)\ x_{1i}(b)$.

REMARK. This lemma and its proof explain the relevance of $EU_n(q)$ for computing in the chunk. The proof will be needed repeatedly.

4.8 NOTATION. If $s \in R$, $v \in R^n$, then $x_{n+2}(v,s)$ stands for $x_{n+2}(v_1,..,v_n,s)$. (cf. [6] 3.12).

DEFINITION Let $q, s \in R$, $v \in EU_n(q)$, $B \in St(\{n+2\} \times \{1\}^*)$, $T \in St(n+1)$, $U \in \underline{Up}$, $w \in R^{n+1}$ such that $x_{n+2}(w)T = Tx_{n+2}(v,s)$.
Then we say that $L(x_{n+2}(w))\ (Tx_{n+2,1}(q)\ B,U)$ is defined and let its value be $L(T)\ L(x_{n+1,n+2}(s))\ \langle X,Y\ B\ U\rangle$, where X,Y are chosen as in lemma 4.7. (Note that $L(x_{n+1,n+2}(s))\ \langle X,Y\ B\ U\rangle$ is defined at the bottom.) To see that $\langle X,Y\ B\ U\rangle$ does not depend on the particular choice of X,Y we may assume $T=1$, $B=1$, $U=1$, $s=0$ and then apply the squeezing principle with $i=n+1$.

REMARKS. Note that this $L(x_{n+2}(w))$ differs from those defined in [6] 3.13, 3.18 in that its argument is in $\underline{Low} \times \underline{Up}$ rather than C.

Of course the idea is to show eventually that $L(x_{n+2}(w))\,(Tx_{n+2,1}(q)B,U)$ depends only on w and $\langle Tx_{n+2,1}(q)\ B,U\rangle$.
Note further that the definition of $L(x_{n+2}(w))(*,*)$ is algorithmic in nature: Given a reduction to zero of v in $EU_n(q)$, the proof of lemma 4.7 tells us how to construct X and Y. In later proofs we will often need to consider the steps in the algorithm. This makes these proofs more tedious than proofs involving only maps defined at the bottom.

.9 <u>DEFINITION</u>. An element s of R is an <u>irrelevant factor</u> when the following holds. For each $q \in R$, $a \in R^n$ with $as \in EU_n(q)$ we have $a \in EU_n(sq)$. It is an easy exercise to show that units are irrelevant factors. As $EU_n(0) = R^n$, zero is also an irrelevant factor. In situation α it is clear that any element of R is an irrelevant factor. In situation β we have the following corollary to the main results of [1], [13], cf. [8]. Intuitively it says that elements in "general position" are irrelevant factors.

<u>LEMMA</u>. (Situation β). There is a non-zero ideal I of R such that, if $s \in R$ is such that s maps to a unit in the semi-local ring R/I, then s is an irrelevant factor.

<u>PROOF</u>. If $(a,b) \in R^2$, $q \in R$ then $(a,b) \in EU_2(q)$ if and only if, in the notations of [8], [13], the column $\begin{pmatrix} 1+aq \\ bq \end{pmatrix}$ occurs as the first column of some element of E(R,qR). If q=0, $EU_2(q) = R^2$. If $q \neq 0$, we learn from [8], [13] that $(a,b) \in EU_2(q)$ if and only if the power norm residue symbol $\left(\frac{1+aq}{bq}\right)_{r(q)}$ vanishes, where r(q) is an integer that depends in a certain way on valuations $v_{p_1}(q),\ldots,v_{p_m}(q)$, where $p_1,\ldots,p_m$ are fixed prime ideals. Now choose for I a non-zero ideal that is divisible by $p_1,\ldots,p_m$. If s is a unit mod I then $v_{p_i}(sq) =$

$v_{p_i}(q)$ so that for $(as,bs) \in EU_2(q)$, $q \neq 0$, we have $\left(\frac{1+asq}{bsq}\right)_{r(q)} = 1$ hence $\left(\frac{1+asq}{bsq}\right)_{r(sq)} = 1$, hence $(a,b) \in EU_2(sq)$.

REMARK. In the sequel we will not use the fact that the condition on s can be satisfied independent of q. It would suffice to know that for any q there are sufficiently many s that are "irrelevant with respect to q".

4.10 LEMMA Let $w \in R^{n+1}$, $v \in R^n$, q, $s \in R$, $T \in St(n+1)$ such that $x_{n+2}(w)T = Tx_{n+2}(v,s)$. Let further $A \in St([n] \times \{n+1\})$, $B \in St(\{n+2\} \times \{1\}^*)$, $U \in \underline{Up}$ and assume that s is an irrelevant factor. If $L(x_{n+2}(w))\ (Tx_{n+2,1}(q)\ B,\ A\ U)$ and $L(x_{n+2}(w))\ (Tx_{n+2,1}(q)\ B\ A,\ U)$ are both defined (see 4.8), then their values are the same.

PROOF. We may move T out of the way (towards the left) and then consider the case T=1. Note that we change w when removing T. In particular, we get $x_{n+2}(w) = x_{n+2}(v,s)$. The reader is expected to take care of such details in the sequel. Say $A = x_{1,n+1}(a_1)\ldots x_{n,n+1}(a_n)$. We have v, $v\text{-}as \in EU_n(q)$. First we want to reduce to the case v=0, arguing by induction on the complexity of (v;q). When we replace B,U by 1, $A^{-1}\ B\ A\ U$ respectively, the answers don't change. Therefore assume B=1. If $z = e^i(t)\ v$ has lower complexity than v, we use the proof of lemma 4.7 to express $L(x_{n+2}(v,s))\ (x_{n+2,1}(q),\ A\ U)$ in terms of $L(x_{n+2}(z,s))\ (x_{n+2,1}(q),\ A^1U^1)$ with suitable A^1,U^1. We find that $L(x_{n+2}(v,s))\ (x_{n+2,1}(q),\ A\ U) = L(x_{1i}(-t))\ L(x_{n+2}(z,s))$ $(x_{n+2,1}(q),\ A^1U^1)$ where $A^1 = x_{1,n+1}(ta_i)\ A$, $U^1 = x_{n+2,i}(-qt)$ $x_{n+2,n+1}(-q\ t\ a_i)\ x_{1i}(t)\ U$. By induction hypothesis we may rewrite the expression as $L(x_{1i}(-t))\ L(x_{n+2}(z,s))\ (x_{n+2,1}(q)\ A^1,U^1)$ and a straightforward computation shows that this equals $L(x_{n+2}(v,s))$

$(x_{n+2,1}(q)\ A,U)$. Similarly, if $z = e_i(t)v$ has lower complexity than v, we use the proof of lemma 4.7 to rewrite $L(x_{n+2}(v,s))$ $(x_{n+2,1}(q), A\ U)$ as $L(x_{i1}(-tq))\ L(x_{n+2}(z,s))\ (x_{n+2,1}(q),\ A\ x_{i,n+2}(-t)\ U)$ which equals (by induction hypothesis) $L(x_{i1}(-tq))\ L(x_{n+2}(z,s))$ $(x_{n+2,1}(q)\ A\ x_{i,n+1}(t\ q\ a_1),\ x_{i,n+1}(-t\ q\ a_1)\ x_{i,n+2}(-t)\ U)$. Note that one has to be careful about what the induction hypothesis tells exactly. One really has to move $A\ x_{i,n+1}(t\ q\ a_1)$ over the comma, not just A. With some patience one now finishes the check for this case too. We may further assume $v=0$. Then we have $-as \in EU_n(q)$ and because s is an irrelevant factor we even have $-a \in EU_n(sq)$. When $a=0$ there is nothing to prove. Therefore let us now argue by induction on the complexity of $(-a;sq)$. We have to show that $L(A)\ L(x_{n+2}(-as,s))$ $(x_{n+2,1}(q),\ x_{n+2,n+1}(q\ a_1)\ U)$ equals $L(x_{n+2,n+1}(s))\ \langle x_{n+2,1}(q),\ A\ U\rangle$. If $z = e^i(t)(-a)$ has lower complexity than $-a$, we argue as before, first rewriting $L(x_{n+2}(-as,s))\ (x_{n+2,1}(q),*)$ in terms of z. If $z = e_i(t)(-a)$ has lower complexity than $-a$, the computations are similar but a little longer. (The i-th co-ordinate of $e_i(t)(-a)$ is $-a_i+t-tsqa_1$).

.11 <u>LEMMA</u>. (Additivity, technical form).
Let $q,r,s \in R$, $v,w \in EU_n(q)$, $U \in \underline{Up}$, $B \in St(\{n+2\} \times \{1\}^*)$, $T \in St(n+1)$, $a,b \in R^{n+1}$ such that $x_{n+2}(a)\ T = T\ x_{n+2}(v,r)$ and $x_{n+2}(b)\ T = T\ x_{n+2}(w,s)$. Say $L(x_{n+2}(a))\ (T\ x_{n+2,1}(q)\ B,U) = \langle P,Q\rangle$ and assume that $L(x_{n+2}(b-a))\langle P,Q\rangle$ is defined at the bottom. In situation β assume further that $v_1=0$ and that $s-r-rqw_1$ is an irrelevant factor. Then $L(x_{n+2}(b))\ (T\ x_{n+2,1}(q)\ B,U) = L(x_{n+2}(b-a))\ L(x_{n+2}(a))$ $(T\ x_{n+2,1}(q)\ B,U)$.

PROOF. As we may move T over to the left and B over to U, we further assume T=1, B=1. We start with reducing to the case v=0. In situation α this is done in the same fashion as in the previous proof, using now also some properties of maps defined at the bottom (cf. [6] lemma 3.23). In situation β we have $v_1=0$ so that the reduction to $v=0$ can be done with steps which do not affect $s-r-rqw_1$. Thus we may reduce to the case $v=0$ in any situation. Put $y=s-r-rqw_1$.
What we have to show boils down to the equality of $L(x_{n+2}(w,s))$ $(x_{n+2,1}(q), U)$ and $L(x_{n+1,1}(rq))\ L(x_{n+2}(w,y))\ \langle x_{n+2,1}(q), x_{n+1,n+2}(r)\ U\rangle$. Of course we may assume U=1. As $L(x_{n+2}(w,y))\langle x_{n+2,1}(q), *\rangle$ is defined at the bottom, there are $d_i \in R$ such that $w_1=d_2w_2+..+d_nw_n+d_{n+1}y$. It is not difficult to see from this that $L(x_{n+2}(w,y))\ (x_{n+2,1}(q)$ $x_{1,n+1}(d_{n+1}), x_{1,n+1}(-d_{n+1})\ x_{n+1,n+2}(r))$ is defined. In fact one can evaluate explicitly in this case and one sees, using [6] lemmas 3.23, 3.24, 3.25, that the result is the same as $L(x_{n+2}(w,y))\ (x_{n+2,1}(q)$ $x_{1,n+1}(d_{n+1}), x_{1,n+1}(-d_{n+1})\ x_{n+1,n+2}(r))$. But we also know, by the previous lemma, that the result equals $L(x_{n+2}(w,y))\ (x_{n+2,1}(q),$ $x_{n+1,n+2}(r))$. Thus it remains to show that $L(x_{n+2}(w,s))\ (x_{n+2,1}(q), 1) =$ $L(x_{n+1,1}(rq))\ L(x_{n+2}(w,y))\ (x_{n+2,1}(q), x_{n+1,n+2}(r))$.
It is easy to see that if the first member in this equation is $\langle Xx_{n+2,1}(q)\ x_{n+1,1}(sq), x_{n+1,n+2}(s)\ Y\rangle$ with $X \in St(n)$, $Y \in St(\{n+1\}^* \times \{1,n+1\}^*)$, then the second member is of the form $\langle x_{n+1,1}(*)\ X\ x_{n+2,1}(q)\ x_{n+1,1}(*), x_{n+1,n+2}(*)\ Y\ x_{n+1,n+2}(*)\rangle$. Using the semi-direct product structure of $St([n+1] \times [n])$ and $St([n+2] \times \{1,n+1\}^*)$, cf. [6] 3.5, we see that the problem reduces to proving an identity of the form $\langle 1,1\rangle = \langle x_{n+1,1}(*), x_{n+1,2}(*)\ldots x_{n+1,n+2}(*)\rangle$. But such identities hold precisely when they hold after applying mat. Now note that at the matrix level the problem has been trivial from the beginning.

4.12 We have to generalize lemma 4.11 to a result like [6] Proposition 3.33. Let us first assume that we are in situation α.

LEMMA (Situation α). Let $v,w \in R^{n+1}$, $X \in$ Low, $Y \in$ Up.
Let $L(x_{n+2}(w))\,\langle X,Y\rangle$ be defined at the bottom with value $\langle P,Q\rangle$ and let $L(x_{n+2}(v))\,\langle P,Q\rangle$ be defined at the bottom.
If further $L(x_{n+2}(v+w))\,(X,Y)$ is defined, then its value is $L(x_{n+2}(v))\,\langle P,Q\rangle$.

PROOF. Say $X = T\, x_{n+2,1}(q)\, B$ with $T \in St(n+1)$, $B \in St(\{n+2\} \times \{1^*\})$.
As usual we move T over to the left and reduce to the case T=1.
By SR^2_{n+1} there is $A \in St([n] \times \{n+1\})$ such that both $L(x_{n+2}(v+w))$ $(X\,A,\ A^{-1}Y)$ and $L(x_{n+2}(w))\,(X\,A,\ A^{-1}Y)$ are defined. Now $L(x_{n+2}(v+w))$ (X,Y) equals $L(x_{n+2}(v+w))\,(X\,A,\ A^{-1}Y)$ by lemma 4.10. Also $L(x_{n+2}(w))$ $(X\,A,\ A^{-1}Y)$ equals $L(x_{n+2}(w))\,\langle X\,A,\ A^{-1}Y\rangle$ as one sees by applying lemma 4.11 with v=0, r=0, a=0. Remains to see that $L(x_{n+2}(v))$ $L(x_{n+2}(w))\,(X\,A^{-1},\ A\,Y) = L(x_{n+2}(v+w))\,(X\,A^{-1},\ A\,Y)$. But this is the situation of lemma 4.11

4.13 DEFINITION. (Situation α). Let $v \in R^{n+1}$, $X \in$ Low, $Y \in$ Up such that mat$(x_{n+2}(v)\ X\ Y) \in$ mat(C). By SR_{n+1} there is $A \in St([n] \times \{n+1\})$ such that $L(x_{n+2}(v))\,(X\,A,\ A^{-1}Y)$ is defined. We put $L(x_{n+2}(v))\,\langle X,Y\rangle =$ $L(x_{n+2}(v))\,(X\,A,\ A^{-1}Y)$. To see that this depends only on $\langle X,Y\rangle$ and v, we note that there is $z \in R^{n+1}$ such that both steps in $L(x_{n+2}(-z))$ $L(x_{n+2}(v+z))\,\langle X,Y\rangle$ are defined at the bottom, because of $SR^2_{n+1}(n+2,n+1)$, as in [6] 3.32. By lemma 4.12 we have $L(x_{n+2}(-z))\ L(x_{n+2}(v+z))\,\langle X,Y\rangle =$ $L(x_{n+2}(v))\,(X\,A, A^{-1}Y)$ and the left hand side is independent of the choices made in the right hand side. (Similarly the right hand side is independent of the choices made in the left hand side, so both sides are independent of choices). It is clear that our present definition of $L(x_{n+2}(v))\,\langle X,Y\rangle$ is compatible with the earlier definitions in 4.8, [6] 3.18 and that it is equivalent with [6] 3.31.

4.14 PROPOSITION. (Additivity in situation α, cf. [6] 3.33).
$L(x_{n+2}(v))\ L(x_{n+2}(w))\ \langle X,Y\rangle = L(x_{n+2}(v+w))\ \langle X,Y\rangle$ whenever the left hand side is defined.

PROOF. First assume the $L(x_{n+2}(v))$ step is defined at the bottom. Use SR^2_{n+1} to choose a representative (P,Q) of $\langle X,Y\rangle$ so that both $L(x_{n+2}(w))$ (P,Q) and $L(x_{n+2}(v+w))$ (P,Q) are defined. Then lemma 4.11 applies. In the general case we want to get back to this special case by perturbation, as in the proof of [6] 3.30. So we seek $z \in R^{n+1}$ such that in $L(x_{n+2}(z))L(x_{n+2}(v))\ L(x_{n+2}(w))\ \langle X,Y\rangle$ and $L(x_{n+2}(v+z))\ L(x_{n+2}(w))$ $\langle X,Y\rangle$ the steps $L(x_{n+2}(z))$, $L(x_{n+2}(v+z))$ are defined at the bottom. This is an SR^2_{n+1} $(n+2,n+1)$ type problem, cf. [6] 3.30, so z exists.

4.15 We now have recovered the results of [6] section 3 in the context of situation α. Therefore we turn to situation β and try to catch up. Note that $n=2$, $n+2=4$. We start with a variation on [6] 3.30. (We will return to situation α in 4.19).

LEMMA. Let $L(x_4(v))\ \langle X,Y\rangle$ be defined at the bottom, with value $\langle P,Q\rangle$. Let $L(x_4(w-v))\ \langle P,Q\rangle$ and $L(x_4(w))\ \langle X,Y\rangle$ be defined firmly at the bottom (cf. 4.5). Then $L(x_4(w))\ \langle X,Y\rangle = L(x_4(w-v))\ L(x_4(v))\ \langle X,Y\rangle$.

PROOF. We may assume $X = x_{41}(q)$, $v_1=0$. First assume $1+qw_1 \neq 0$. We want to modify the situation so that lemma 4.11 applies. To adapt v,w we will choose a suitable $M \in St(\{2,3\} \times \{2,3\})$, multiply the desired equality from the left by $L(M)$ and then move M over to Y. As $L(x_4(w))\ \langle X,Y\rangle$ is defined firmly at the bottom, the column (w_2,w_3) is unimodular. Similarly the column $(w_2-v_2-v_2q\ w_1, w_3-v_3-v_3\ q\ w_1)$ is unimodular. The effect of pushing M through is to transform these two unimodular columns by an element of $E_2(R)$. It is not difficult to see, using the Chinese Remainder Theorem, that M can be chosen such that the new w_3 has an invertible image in R mod $(1+q\ w_1)$ and the new w_3, $w_3-v_3-v_3\ q\ w_1$ are irrelevant factors. (cf. 4.9). Thereafter we can modify further, using an M of the form $x_{23}(*)$, so that w_2-1 becomes a multiple of $1+q\ w_1$. It follows from lemma 4.11 with T=1, a=0, that $L(x_4(w))\ \langle X,Y\rangle = L(x_4(w))\ (X,Y)$. Also it is clear (cf. [6] lemma 3.25) that $L(x_4(v))\ \langle X,Y\rangle = L(x_4(v))\ (X,Y)$. We may thus finish by lemma 4.11. Remains the case that $1+q\ w_1 = 0$. We may assume $v_2=0$. Now if $w_2=0$ the result follows from the squeezing principle with i=2. If $w_2 \neq 0$ we can get back to the case $1+q\ w_1 \neq 0$ by pushing through $M=x_{12}(1)$ in the same fashion as above.

4.16 LEMMA. Let both steps in $L(x_4(u-v))\ L(x_4(v))\ \langle X,Y\rangle$ and both steps in $L(x_4(u-w))\ L(x_4(w))\ \langle X,Y\rangle$ be defined firmly at the bottom. Then the end results agree.

PROOF. First we note that if $L(x_4(y))\ \langle P,Q\rangle$ is defined (firmly) at the bottom, then the same holds for the other step in $L(x_4(-y))\ L(x_4(y))\ \langle P,Q\rangle$. (And the result is $\langle P,Q\rangle$, of course). As in the proof of [6] 3.30 we look for $z \in R^3$ such that $L(x_4(z))\ L(x_4(u-v))\ L(x_4(v))\ \langle X,Y\rangle = L(x_4(z+u-v))\ L(x_4(v))\ \langle X,Y\rangle = L(x_4(z+u-v))\ L(x_4(-z-u+v))\ L(x_4(z+u))\ \langle X,Y\rangle = L(x_4(z+u))\ \langle X,Y\rangle = \cdots = L(x_4(z+u-w))\ L(x_4(w))\ \langle X,Y\rangle = L(x_4(z))\ L(x_4(u-w))\ L(x_4(w))\ \langle X,Y\rangle$. By the previous lemma, 4.6, [6] 3.22, and the remark above, this means that we want the steps $L(x_4(z))$, $L(x_4(z+u-v))$, $L(x_4(z+u-w))$ to be defined firmly at the bottom and the step $L(x_4(z+u))$ defined at the bottom. This amounts to four conditions on z and that is a lot, so we have to analyze closely what they look like. As usual we may assume $X = x_{41}(*)$. As in [6] 3.30 there are three 2 by 3 matrices, say A_1,A_2,A_3, each completable to invertible 3 by 3 matrices, and three vectors a_1,a_2,a_3 in R^2 such that the first three conditions are equivalent to unimodularity of the $a_i+A_i\,z$. To satisfy the fourth condition we need to take z of the form $-u+y\,r$ with $y \in R^3$, (y_2,y_3) unimodular, $r \in R$. So there are four vectors that have to be unimodular: $a_i-A_iu+A_iyr$ (i=1,2,3) and (y_2,y_3). From the considerations in [6] section 2 (cf. [6] 2.11) we see that we can satisfy these requirements provided there is $r \neq 0$ such that for each maximal ideal $\underline{m}$ the conditions $a_i-A_iu+A_iyr \not\equiv 0 \mod \underline{m}\, R^2$, $(y_2,y_3) \not\equiv 0 \mod \underline{m}\, R^2$ can be met simultaneously. ("Local solvability implies global solvability"). For a given $\underline{m}$ the existence of a local solution for y clearly only depends on the vanishing or non-vanishing of r mod $\underline{m}$. By a count as in [6] 2.9 we see that the $\underline{m}$ where there is no solution with non-vanishing r mod $\underline{m}$ are of a very particular type: For such $\underline{m}$ we have $R/\underline{m} \cong \mathbb{F}_2$ and $a_i-A_iu \not\equiv 0 \mod \underline{m}\, R^2$. So at these $\underline{m}$ there is a solution with vanishing r mod $\underline{m}$. If no such $\underline{m}$ exists, take r=1. In the contrary case observe that none of the a_i-A_iu vanishes identically so that there are only finitely many $\underline{m}$ where one needs to keep r mod $\underline{m}$ from vanishing . Therefore r can be chosen suitably.

4.17 LEMMA. Let $v,w \in R^3$, $\langle X,Y \rangle \in C$ such that both $\underline{mat}(x_4(w)XY)$ and $\underline{mat}(x_4(v+w)XY)$ are in $\underline{mat}(C)$. Then there are $t,u \in R^3$ such that all steps in $L(x_4(t))L(x_4(v-t)L(x_4(u))L(x_4(w-u))\langle X,Y \rangle$ and in $L(x_4(t))L(x_4(v+w-t))\langle X,Y \rangle$ are defined firmly at the bottom. Moreover, the end results agree.

PROOF. By [6] 2.11 condition $SR_3^3(4,3)$ is satisfied. Therefore existence of t,u follows as in [6] 3.32. Now substitute w-u for v, w for u, v+w-t for w in Lemma 4.16.

4.18 DEFINITION. (Compare [6] 3.31). Let $v \in R^3$, $\langle X,Y \rangle \in C$ such that $\underline{mat}(x_4(v)XY) \in \underline{mat}(C)$. There is $z \in R^3$ such that both steps in $L(x_4(-z))L(x_4(v+z))\langle X,Y\rangle$ are defined firmly at the bottom. (cf. 4.17). We define $L(x_4(v))\langle X,Y\rangle$ to be equal to the result. It follows from Lemma 4.16 that this does not depend on the choice of z. When $L(x_4(v))\langle X,Y\rangle$ is defined at the bottom, the present definition is consistent with the one in [6] 3.18 by Lemma 4.15. Further "additivity" (cf. [6] 3.33) holds by Lemma 4.17. It easily follows that the present definition is also consistent with [6] 3.13, 3.15, 3.31. Now read [6] 3.34. By induction on complexity one easily shows that $L(x_4(v))\langle X,Y\rangle$ equals $L(x_4(v))(X,Y)$ when the latter is defined.

.19 We have recovered the results of [6] section 3 both for situation α and for situation β. Therefore let us look at [6] section 4. Read [6] 4.1 through 4.7. Note that the computation in [6] 4.6 no longer looks horrendous when compared with the present paper. Our next task is to prove an analogue of [6] Proposition 4.9. First we consider situation α.

<u>LEMMA</u>. (Situation α). Let $v \in R^{n+1}$, $t \in R$, $A \in St(\{n+2\} \times [n+2])$, $B \in St(\{1\} \times [n+2])$ such that $R(x_{21}(t))L(x_{n+2}(v))\langle A,B\rangle$ is defined. Then it equals $L(x_{n+2}(v))R(x_{21}(t))\langle A,B\rangle$.

<u>PROOF</u>. Note that <u>mat</u>$(ABx_{21}(t))$ and <u>mat</u>$(x_{n+2}(v)ABx_{21}(t))$ are in <u>mat</u>(C), so that $L(x_{n+2}(v))R(x_{21}(t))\langle A,B\rangle$ is indeed defined. Write it as LRp and the other version as RLp. (So $p = \langle A,B\rangle$ etc.) It suffices to show that there are $T \in St([n] \times \{n+1\})$, $U \in St(\{1\}^* \times \{n+1\})$ such that $R(U)L(T)LRp = R(U)L(T)RLp$. Because of SR^2_{n+1} we can choose

T,U in such a way that, by pushing $R(U)L(T)$ over to p, we are left with a version of the original problem in which the following holds. Both L_1L_2Rp and RL_1L_2p are defined, where $L_1 = L(x_{n+1,n+2}(v_{n+1}))$, $L_2 = L(x_{n+2}(v_1,\dots,v_n,0))$. The squeezing principle, with i = n+1 implies that $L_2Rp = RL_2p$, because 4.8 gives a method to evaluate both sides "away from line i+1". Remains to show that $L_1RL_2p = RL_1L_2p$. Using what we know about the shape of L_2p and applying inv, we see that it boils down to the case that, in the original context, v_2 vanishes. Write $A = x_{n+2,1}(q)x_{n+2,2}(a_2)\dots x_{n+2,n+1}(a_{n+1})$. We now wish to argue by induction on the complexity of $(v_1,v_3,\dots,v_{n+1})$ with respect to q. The case v = 0 is obvious. When $(v_1+rv_i,v_3,\dots,v_{n+1})$ has lower complexity, simply multiply by $L(x_{1i}(r))$ and push it through. Remains the case that $(v_1,v_3,\dots,v_i+r(1+qv_1),\dots,v_{n+1})$ has lower complexity, $3 \leqslant i \leqslant n+1$. Then $RLp =$

$$RL(x_{i2}(-ra_2)x_{i1}(-rq))R(x_{i,n+2}(-r)x_{n+2,3}(a_3)\dots x_{n+2,n+1}(a_{n+1}))$$
$$L(x_{n+2}(v_1,0,v_3,\dots,v_i+r(1+qv_1),\dots,v_{n+1}))$$
$$\langle x_{n+2,1}(q)x_{n+2,2}(a_2),x_{12}(*)\dots x_{1,n+2}(*)\rangle$$

. Now R can be pushed through all the maps in the right hand side. It follows that $L^{-1}RLp = L(x_{i2}(-ra_2)x_{i1}(-rq))$ $R(x_{i,n+2}(-r)x_{n+2,3}(a_3)\dots x_{n+2,n+1}(a_{n+1}))L(x_{i,n+2}(r))R\langle x_{n+2,1}(q)$ $x_{n+2,2}(a_2),x_{12}(*)\dots x_{1,n+2}(*)\rangle$. In this last expression we may move R to the left because $L(x_{i,n+2}(r))$ and $R = R(x_{21}(t))$ slide past each other here. (cf. [6] 4.5, 4.6). It easily follows that $R^{-1}L^{-1}RLp$ equals p.

4.20 PROPOSITION. (Situation α). (cf. [6] 4.9). Let $v = (v_1,\dots,v_{n+1})$, $w = (w_2,\dots,w_{n+2})$. Then $L(x_{n+2}(v)) \circ R(x_1(w)) \approx R(x_1(w)) \circ L(x_{n+2}(v))$.

PROOF. Let both composites be defined at $p = \langle X,Y \rangle$. We have to show that the values agree. When $L(x_{n+2}(v))\,p$ is defined at the bottom we apply inv and get essentially into the situation of Lemma 4.19. In general $SR^2_{n+1}\,(n+2,n+1)$ implies that there is $z \in R^{n+1}$ such that $L(x_{n+2}(z))$ is defined at the bottom at $R(x_1(w))L(x_{n+2}(v))p$ and $L(x_{n+2}(v+z))$ is defined at the bottom at p. Writing $L_1 = L(x_{n+2}(v))$, $L_2 = L(x_{n+2}(z))$, $R = R(x_1(w))$ we get $L_1Rp = L_2^{-1}L_2L_1Rp = L_2^{-1}RL_2L_1p = RR^{-1}L_2^{-1}RL_2L_1p = RL_2^{-1}R^{-1}RL_2L_1p = RL_1p$, where in the fourth equality we use that L_2^{-1} is defined at the bottom at RL_2L_1p.

.21 We have to prove the analogue of this proposition for situation β too. This is more complicated.

NOTATION. (situation β). When $X \in St(4)$ let us write $\underline{mat}_{ij}(X)$ for the entry of mat(X) at the intersection of the i-th row and the j-th column. So $\underline{mat}_{41}(X)$ denotes the entry in the lower left hand corner.

LEMMA. (situation β). Let $A = x_{41}(q)x_{42}(r)$, $B = x_{13}(*)x_{14}(*)$, $v \in R^3$, $t \in R$, $Z = \underline{inv}(x_4(v)A)$, such that the pairs (q,r) and $(\underline{mat}_{11}(Z),\underline{mat}_{21}(Z))$ are unimodular. Then $L(x_4(v))R(x_{21}(t))\langle A,B \rangle = R(x_{21}(t))L(x_4(v))\langle A,B \rangle$. (Both sides are defined.)

PROOF. To see that $R(x_{21}(t))L(x_4(v))\langle A,B \rangle$ is defined, one inspects the first column of mat $\underline{inv}(x_4(v)ABx_{21}(T))$. Consider the unimodular pair $(1+qv_1+rv_2,v_3)$. When it equals $(0,1)$ the result follows by applying inv to [6] 4.7. Now let V be the set of unimodular pairs such that the lemma holds

whenever $(1+qv_1+rv_2,v_3) \in V$. By lemma 4.3 it suffices to show that V is invariant under $E_2(R)$. Multiplying our problem from the left by $L(x_{13}(a)x_{23}(b))$, $R(x_{23}(-b)x_{43}(*))$ and pushing these two maps over to $\langle A,B\rangle$ we can replace (v_1,v_2) by (v_1+av_3,v_2+bv_3), hence $(1+qv_1+rv_2,v_3)$ by $(1+qv_1+rv_2+(qa+rb)v_3,v_3)$, for any $a,b \in R$. As (q,r) is unimodular this means that V is closed under the operation $(f,g) \mapsto (f+cg,g)$ for any $c \in R$. Further, for $a \in R$ one has $L(x_{34}(a))R(x_{12}(t))\langle A,B\rangle =$ $R(x_{12}(t))L(x_{34}(a))\langle A,B\rangle$ by [6] 4.6. This means that our problem is equivalent to showing that
$R(x_{12}(t))L(x_4(v_1,v_2,v_3-a))\langle P,Q\rangle$ equals
$L(x_4(v_1,v_2,v_3-a))R(x_{12}(t))\langle P,Q\rangle$ with $\langle P,Q\rangle = L(x_{34}(a))\langle A,B\rangle$.
But this last problem boils down to one of the original type with v_3 replaced by $v_3-a-aqv_1-arv_2$. So V is also closed under the operation $(f,g) \to (f,g+af)$.

.22 We say that $R(x_1(w))\langle X,Y\rangle$ is defined (firmly) at the bottom when $L(\underline{\text{inv}}(x_1(w)))$ is defined (firmly) at the bottom at $\underline{\text{inv}}\langle X,Y\rangle$.

<u>LEMMA</u>. (Situation β). Let v,w,X,Y be as usual and assume that $L(x_4(v))\langle X,Y\rangle$ is defined, while $R(x_1(w))$ is defined firmly at the bottom at both $\langle X,Y\rangle$ and $L(x_4(v))\langle X,Y\rangle$. Assume also that $(\underline{\text{mat}}_{41}(XY),\underline{\text{mat}}_{41}(XYx_1(w)))$ is a unimodular pair. Then $R(x_1(w))L(x_4(v))\langle X,Y\rangle = L(x_4(v))R(x_1(w))\langle X,Y\rangle$.

<u>PROOF</u>. We may assume $x_1(w) = x_{21}(t)$, $X = x_{41}(q)x_{42}(r)$, $q,r,t \in R$, $Y = x_{13}(*)x_{14}(*)$. Note that (q,r) is unimodular because $\underline{\text{mat}}_{41}(XY) = q$, $\underline{\text{mat}}_{41}(XYx_1(w)) = q+rt$. Applying <u>inv</u> to Lemma 4.6 we see that all conditions of Lemma 4.21 are satisfied.

23 PROPOSITION. (Situation β). (cf. 4.20). Let v,w,X,Y be as usual and assume that both $L(x_4(v))R(x_1(w))\langle X,Y\rangle$ and $R(x_1(w))L(x_4(v))\langle X,Y\rangle$ are defined. Then their values agree.

PROOF.

CASE 1. $X \in St(\{4\} \times [4])$, $Y \in St(\{1\} \times [4])$, $w = (t,0,0)$, both $\underline{mat}_{41}(XY)$ and $\underline{mat}_{41}(XYx_{21}(t))$ are non-zero and v_3 is in the intersection of those maximal ideals $\underline{m}$ for which $R/m \cong \mathbb{F}_2$. We claim there is z such that $R(x_1(z))L(x_4(v))R(x_{21}(t))\langle X,Y\rangle$ equals $L(x_4(v))R(x_1(z))R(x_{21}(t))\langle X,Y\rangle$ and $R(x_1(z+w))L(x_4(v))\langle X,Y\rangle$ equals $L(x_4(v))R(x_1(z+w))\langle X,Y\rangle$. By Lemma 4.22 this creates an $SR_3^4(4,3)$ type problem, except that one also needs to make $\underline{mat}_{41}(XYx_1(z+w))$ prime to the product of $\underline{mat}_{41}(XY)$ and $\underline{mat}_{41}(XYx_{21}(t))$. As before the problem of finding z is solvable (globally) if it is solvable locally (cf. proof of 4.16 and proof of Theorem 3 in [6] section 2.) Where the residue field has at least three elements there is an easy count as in [6] 2.9. At a place with a residue field with 2 elements one can check that there is a solution of the form $(z_2,z_3,z_4) = (z_2,1,z_4)$. (Apply $\underline{inv}$ to Lemma 4.6).

CASE 2. X,Y,w as in case 1 and both $\underline{mat}_{41}(XY)$ and $\underline{mat}_{41}(XYx_{21}(t))$ are non-zero. We wish to get back to case 1. We may assume $X = x_{41}(q)x_{42}(r)$, $Y = x_{13}(*)x_{14}(*)$. From the proof of Lemma 4.21 we see that we may replace $(v_1,v_2,v_3;q,r,t)$ by $(v_1+av_3,v_2+bv_3,v_3;q,r,t)$ for any $a,b \in R$. Therefore we may assume qv_1+rv_2 is contained in each maximal ideal $\underline{m}$ with $R/\underline{m} \cong \mathbb{F}_2$, $v_3 \notin \underline{m}$. But from the same proof we see that we may replace $(v_1,v_2,v_3;q,r,t)$ by $(v_1,v_2,v_3-v_3(1+qv_1+rv_2);q,r,t)$. Therefore we may reduce to case 1 indeed.

<u>CASE 3</u>. Both $\underline{\text{mat}}_{41}(XY)$ and $\underline{\text{mat}}_{41}(XYx_1(w))$ are non-zero. The proof is similar to the proof of case 1: First one notes that the special case in which $R(x_1(w))\langle X,Y\rangle$ is defined at the bottom is essentially the same as case 2. Then one multiplies by a suitable $R(x_1(z))$ and applies this special case twice. (To prove the existence of a suitable z pick a maximal ideal $\underline{m}$ for which $R/\underline{m}$ has at least a hundred elements, say, and replace the requirement $\underline{\text{mat}}_{41}(XYx_1(w)x_1(z)) \neq 0$ by the stronger requirement $\underline{\text{mat}}_{41}(XYx_1(w)x_1(z)) \not\equiv 0 \bmod \underline{m}$. Then argue as in case 1.)

<u>CASE 4</u>. $L(x_4(v))$ is defined at the bottom at both $\langle X,Y\rangle$ and $R(x_1(w))\langle X,Y\rangle$. We may assume that X,Y are as in case 1 and that $v = (0,0,t), t \in R$. By [6] Lemma 4.7 we have $R(x_1(1,a,b)) L(x_4(v))\langle X,Y\rangle = L(x_4(v))R(x_1(1,a,b))\langle X,Y\rangle$ for all $a,b \in R$. Therefore it is easy to reduce to the case that $\underline{\text{mat}}_{41}(XY)$ is non-zero, without changing $\underline{\text{mat}}_{41}(XYx_1(w))$. Similarly one may also make $\underline{\text{mat}}_{41}(XYx_1(w))$ non-zero so that case 3 applies.

<u>CASE 5</u>. The general case. We choose z such that $L(x_1(z))$ is defined firmly at the bottom at both $R(x_1(w))L(x_4(v))\langle X,Y\rangle$ and $L(x_4(v))\langle X,Y\rangle$, while $L(x_4(z+v))$ is defined firmly at the bottom at both $\langle X,Y\rangle$ and $R(x_1(w))\langle X,Y\rangle$. To see that this can be done one argues as usual, noting in this case that when k is a field one can not fill all of k^3 with four lines of which at least two pass through the origin. (cf. [6] 2.9). The rest is easy. (cf. [6] proof of 4.9).

.24 Now that we have proved the analogue of [6] 4.9 in both situation α and situation β, nothing prevents us from using the remainder of [6] section 4. Therefore $K_2(n+1,R) \to K_2(n+2,R)$ is injective. More generally we see that $K_2(m,R) \to K_2(R)$ is injective for $m \geq n+1$. (Recall that enlarging the size only makes things better, cf. 3.3., 3.4). The remainder of Theorems 1 and 2 easily follows from the following.

LEMMA. Let $x \in St(n+1)$ such that the first row and the first column of mat(x) are trivial. Then there is $g \in St(\{1,n+2\}^* \times \{1,n+2\}^*)$ which equals x in St(n+1).

PROOF. Let τ be the isomorphism $St(n+1) \to St(\{1\}^* \times \{1\}^*)$ obtained by substitution of n+2 for the index 1. In St(n+2) we have $\tau(x) = w_{n+2,1}(1)xw_{n+2,1}(1)^{-1} = x$.(cf. [6] proof of 3.10). For $U \in St(\{1\}^* \times \{1\}^*)$ it is easy to see that $\rho(U)\langle 1,1\rangle = \langle 1,U\rangle$. ($\rho$ as in [6] 4.16). So on the one hand $\rho(x)\langle 1,1\rangle = \langle x,1\rangle$ and on the other hand $\rho(x)\langle 1,1\rangle = \rho(\tau(x))\langle 1,1\rangle = \langle 1,\tau(x)\rangle$. This means that there is $g \in$ Med with $(1,\tau(x)) = (xg^{-1},g)$. Now recall the semi-direct product structure of Med and apply mat to see that g comes from $St(\{1,n+2\}^* \times \{1,n+2\}^*)$.

§5 A simpler proof of surjectivity in Theorem 1.

5.1 In the previous section surjectivity came as a byproduct of our proof of injectivity. For those readers who are mainly interested in surjectivity of the map $K_2(2,R) \to K_2(R)$, we now give a proof that is considerably simpler. (R as in Theorem 1).
The idea is to sharpen a proof of ordinary surjective stability for K_2 (we use something like the "transpose" of the proof in [12]). The sharpening is achieved by means of Vasertein's result that $EU_2(q) = U_2(q)$ for suitable q. (cf. 2.7).

5.2 Recall that it suffices to show that $K_2(2,R) \to K_2(3,R)$ is surjective. Therefore we try to write elements of St(3,R) in normal form. As normal form we take the one suggested by 3.5 and [6] 3.36: We say that X can be written in normal form if there are $L_1 \in$ image ($\underline{Low} \to St(3)$), $U \in$ image ($\underline{Up} \to St(3)$), $L_2 \in$ image ($St([3] \times \{1\}) \to St(3)$) such that $X = L_1 U L_2$. (Notations of [6] 3.4, 3.5, 3.6 with n = 1).
Note that if $X \in St(3)$ can be written in normal form and $L \in$ image ($\underline{Low} \to St(3)$), the element LX can also be written in normal form. Similarly, if X can be written in normal form and $L \in$ image ($St(\{1\}^* \times [3]) \to St(3)$), then XL can be written in normal form.
Let P(a,b,q,r,s) denote the property: $x_{13}(a)\ x_{23}(b)\ x_{31}(q)\ x_{32}(r)\ x_{13}(s)$ can be written in normal form. Let Q(a,b,q,r,s,t) denote the property: $x_{13}(a)\ x_{23}(b)\ x_{31}(q)\ x_{32}(r)\ x_{12}(t)\ x_{13}(s)$ can be written in normal form. Note that Q(a,b,q,r,s,0) = P(a,b,q,r,s).

5.3 PROPOSITION. Let $Y = x_{13}(a)\ x_{23}(b)\ x_{31}(q)\ x_{32}(r)\ x_{12}(t)\ x_{13}(s)$ with a,b,q,r,s,t, $\in$ R. Then Y can be written in normal form.

PROOF. We have to prove Q(a,b,q,r,s,t).

STEP 1 P(a-tb,b,q,r+qt,s) $\Leftrightarrow$ Q(a,b,q,r,s,t).

This one sees by multiplying Y from the left by $L = x_{12}(-t) \in$ image (Low $\to$ St(3)) and making some obvious computations. (It helps to write down the corresponding matrices).

STEP 2. $Q(a,b,q,*,*,*) \Leftrightarrow Q(a+pb,b,q,*,*,*)$, for any p.

To prove step 2, multiply Y from the left by $x_{12}(p)$. Etcetera.

STEP 3. $Q(a,b,q,*,*,*) \Leftrightarrow Q(a,b+p(1+qa),q,*,*,*)$, for any p.

To see this, multiply Y from the left by $x_{21}(pq)$, form the right by $x_{32}(-r)\ x_{23}(p)$.

STEP 4. $Q(a,b,q,r,s,t)$ holds if $(a,b) \in EU_2(q)$.

For, by the previous two steps we may assume $(a,b) = (0,0)$. Then it is obvious. (Compare also 4.7).

NOTATION. In the totally imaginary case (cf. 2.7) let m denote the order of the group of roots of unity in R. Otherwise let m be any non-zero non-unit in R.

STEP 5. $P(a,b,q,*,*)$ holds if q is prime to m and $(1+qa,b)$ is unimodular.

Clearly $(a,b) \in U_2(q)$. As q is prime to m we may apply Vaserstein's theorem which tells us that $U_2(q) = EU_2(q)$. (Cf. 2.7, [13], [1]).

STEP 6. $P(a,b,q,r,s) \Leftrightarrow P(a,b-pa,q+rp,r,s)$, for any p.

To see this, multiply Y from the left by $x_{21}(-p)$, from the right by $x_{21}(p)\ x_{23}(ps)$. (Of course $t = 0$ now).

STEP 7. $P(a,b,q,r,s,) \Leftrightarrow P(a-spb,b,q,r+(1+qs)p,s)$, for any p.

Here we multiplied Y from the left by $x_{12}(-sp)$, from the right by $x_{32}(p)$.

STEP 8. $P(a,b,q,r,s) \Leftrightarrow P(a+sr(b+p+pqa),b+p+pqa,q,-qsr,s-srp)$

By step 1 it suffices to prove $Q(a,b,q,r,s,0) \Leftrightarrow Q(a,b+p+pqa,q,0,s-srp,-sr)$. This is done as in step 3.

STEP 9. $P(a,b,q,r,s)$ holds if q,b are prime to m and $1+qa+rb$ is non-zero.

To see this, note that $(1+qa+rb,b,ma)$ is unimodular, with the first entry non-zero, so that there in $y \in R$ such that $(1+qa+rb,b-yma)$ is unimodular. Then $(1+qa+rb-r(b-yma),\ b-yma) = (1+qa+ryma,\ b-yma)$ is also unimodular and $P(a,b-yma,\ q+rym,r,s)$ holds by step 5. Now apply step 6.

STEP 10. $P(a,b,q,r,s)$ holds if q,b are prime to m.

As b is not zero we can use step 7 to reduce to the situation of step 9.

STEP 11. $P(a,b,q,r,s)$ always holds.

Because of the previous step, we wish to get b,q prime to m. This is a local problem, therefore not difficult. We apply step 7 to get r prime to q, then step 6 to get q prime to m. Via step 8 we can make s trivial modulo the primes $\underline{m}$ that divide m but not r. Repeating step 8 we arrive at the situation that some power of r is divisible by m. We still have q prime to m. Computing modulo primes that divide m we easily see that we can get b prime to m by means of steps 6 and 8, while keeping q prime to m. Because of step 1 the proposition follows.

5.4 PROPOSITION. Any element of St(3) can be written in normal form.

PROOF. Let V be the set of elements that can be written in normal form. We want to show that V is invariant under left multiplication by Steinberg generators. The difficult case is left multiplication by $x_{1i}(a)$, $i = 1$ or 2. Given $X \in V$ we need to prove $x_{1i}(a)\, X \in V$. Multiplying from the left by elements from St(2), from the right by elements from $St(\{1\}^* \times [3])$, and using the semi-direct product structures of <u>Up</u>, <u>Low</u> (cf. [6] 3.5), one easily reduces to the previous proposition.

5.5 Let $\tau \in K_2(3,R)$. We want to show that τ comes from $K_2(2,R)$. Write τ in normal form $L_1\, U\, L_2$. Replacing τ by $L_2\ \tau\ L_2^{-1}$ we reduce to the case $\tau = L_1 U$. As <u>mat</u>(L_1) = <u>mat</u>(U^{-1}) there are $a,b \in R$ with <u>mat</u>$(L_1) = e_{12}(a)\ e_{32}(b)$. Pushing a factor $x_{12}(a)\ x_{32}(b)$ over from L_1 to U, we get to the situation that L_1, U themselves are in $K_2(3,R)$. Because of the semi-direct product structure of <u>Low</u> the element L_1 must come for St(2,R) (use <u>mat</u>), hence from $K_2(2,R)$. Similarly U comes from $St(\{1\}^* \times \{1\}^*)$, hence lies in a conjugate of the (central) image of $K_2(2,R)$, hence comes from $K_2(2,R)$.

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