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Generators and Relations in Algebraic K-Theory

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Despite the transformation of algebraic K-theory by the introduction of higher algebraic K-theory, it still makes sense to look at matrices in order to get a better understanding of K_1 and K_2. I will discuss a sample of results in which this classical approach plays a role. If anything, this sample should give a fair idea of my own interests. For a more balanced overview of algebraic K-theory and some motivating background I may refer to the proceedings of the two previous International Congresses. (See the talks of Quillen, Bass and Gersten at Vancouver and the talks of Swan, Tate, Karoubi at Nice.)

The approach I have in mind can be illustrated with the Bass–Milnor–Serre solution of the congruence subgroup problem for SL_n. This is the problem to decide if each subgroup of finite index in $\mathrm{SL}_n(\mathcal{O})$ contains a subgroup $\mathrm{SL}_n(\mathcal{O}, I)=\ker\left(\mathrm{SL}_n(\mathcal{O})\to\mathrm{SL}_n(\mathcal{O}/I)\right)$ for some ideal I of $\mathcal{O}$ when $\mathcal{O}$ is, say, the ring of integers in a number field. To answer this question (for $n\geqslant 3$) they had to compute the relative K-group $SK_1(\mathcal{O}, I)$ for every ideal I of $\mathcal{O}$. (Definitions of K_1 and K_2 groups will be recalled below.) The computation of $SK_1(\mathcal{O}, I)$ involved several steps. First a *stability theorem* was proved stating that the stabilization maps $SK_1(r, \mathcal{O}, I)\to SK_1(\mathcal{O}, I)$ are surjective for $r\geqslant 2$ and injective for $r\geqslant 3$. Next the *prestabilization* problem was solved, i.e. generators were given for the kernel R of $SK_1(2, \mathcal{O}, I)\to SK_1(3, \mathcal{O}, I)$. By choosing generators and relations for $SK_1(2, \mathcal{O}, I)/R$, which is thus isomorphic to $SK_1(\mathcal{O}, I)$, a *presentation* for $SK_1(\mathcal{O}, I)$ was then obtained, the presentation by Mennicke symbols and their "universal" relations. *Test maps* were found (with values in the group of roots of unity in $\mathcal{O}$), yielding lower bounds for $SK_1(\mathcal{O}, I)$. Finally, the arithmetic of the ring was further exploited to *compute* $SK_1(\mathcal{O}, I)$ exactly. Thus, finding the presentation for $SK_1(\mathcal{O}, I)$ was an important step, but it was by no means the final step. I will ignore this observation and mainly look at stability for K_1 and K_2 and presentations for K_2. I should remark that if stability sets in later than in the situation above, one tends to get less concrete information, when trying the same approach.

1. Basic notions. Let A be a ring (always associative with unit). We embed the group $\mathrm{GL}_n(A)$ into $\mathrm{GL}_{n+r}(A)$ by means of the *stabilization map* $M \mapsto \left(\begin{smallmatrix} M & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where 1 is the identity in $\mathrm{GL}_r(A)$. The direct limit or union of the $\mathrm{GL}_n(A)$ we call $\mathrm{GL}(A)$ or $\mathrm{GL}_\infty(A)$, the stable general linear group. For $a \in A$, $i \neq j$, the *elementary matrix* $e_{ij}(a)$ has ones on the diagonal, a at the intersection of the ith row and the jth column, and zeroes elsewhere. The subgroup of $\mathrm{GL}_n(A)$ generated by the elementary matrices is called $E_n(A)$, the elementary subgroup. We again have stabilization maps $E_n(A) \to E_{n+r}(A)$ and we put $E(A) = E_\infty(A) = \varinjlim_n E_n(A)$. It turns out that $E(A) = [\mathrm{GL}(A), \mathrm{GL}(A)]$ and we put $K_1(A) = \mathrm{GL}(A)/E(A)$, which is thus abelian. A *transvection* in $\mathrm{GL}_n(A)$ is a linear transformation of the form $1 + vaw$ where $1 = \mathrm{id}$, v is a column of length n, $a \in A$, w is a row of length n with $wv = 0$, and w is *unimodular* (i.e. there is a column y with $wy = 1 \in A$). Let $T_n(A)$ denote the subgroup of $\mathrm{GL}_n(A)$ generated by transvections. More generally, if I is a two-sided ideal in A, let $E_n(A, I)$ be the smallest normal subgroup of $E_n(A)$ containing the elementary matrices $e_{ij}(t)$ with $t \in I$, and let $T_n(A, I)$ be the group generated by the transvections $1 + vaw$ with $a \in I$. (So $T_n(A, I)$ contains $E_n(A, I)$.) If $n \geq 3$, then $E_n(A, I)$ is generated by the $e_{ji}(a) e_{ij}(t) e_{ji}(-a)$ with $t \in I$, $a \in A$, and, as always, $i \neq j$ (cf. [3, Appendix 1]). If, moreover, A is *almost commutative* (i.e. finitely generated as a module over its center), Suslin has shown by a localization technique that $E_n(A, I) = T_n(A, I)$, so that $E_n(A, I)$ is a normal subgroup of $\mathrm{GL}_n(A)$. (This usually fails for $n = 2$, even if $A = I$.) We put $K_1(n, A) = \mathrm{GL}_n(A)/E_n(A)$. This pointed set is thus often a group, though not always abelian. (I have been told that it is not abelian for $n = 15$ when A is the ring of continuous real valued functions on the product of two 7-spheres.) The stabilization maps for the GL_m and E_m induce stabilization maps $K_1(n, A) \to K_1(n+r, A)$. Note that such a map is injective if and only if $\mathrm{GL}_n(A) \cap E_{n+r}(A) = E_n(A)$. Similarly we have $K_1(n, A, I) = \mathrm{GL}_n(A, I)/E_n(A, I)$, where $\mathrm{GL}_n(A, I) = \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(A/I))$, and we write $K_1(A, I)$ for $K_1(\infty, A, I)$. If A is commutative, the group $SK_1(n, A, I)$ is the analogue of $K_1(n, A, I)$ with GL_n replaced by SL_n.

While K_1 measures when matrices differ by a product of elementary matrices, K_2 measures those relations between elementary matrices which depend on the ring. (And K_3 measures relations between relations, cf. [3], as is illustrated nicely by K. Igusa's recent concrete description of an element of order 16 in $K_3(\mathbf{Z})$. So the approach with generators and relations even seems to penetrate K_3 a little.) For $n \geq 3$ the Steinberg group $\mathrm{St}_n(A)$ is defined by the following presentation. Take a generator $x_{ij}(a)$ for each $e_{ij}(a)$ in $E_n(A)$. Take as defining relations the following universal relations between elementary matrices (the *Steinberg relations*)

$x_{ij}(a)x_{ij}(b)=x_{ij}(a+b)$; $[x_{ij}(a), x_{jk}(b)]=x_{ik}(ab)$; $[x_{ij}(a), x_{kl}(b)]=1$ when $j\neq k$, $i\neq l$. There is an obvious map from $\mathrm{St}_n(A)$ onto $E_n(A)$ and its kernel is called $K_2(n, A)$. (For $n=2$ one more type of relation must be added to the list.) As usual we have stabilization maps and we write $\mathrm{St}(A)=\mathrm{St}_\infty(A)$, $K_2(A)=K_2(\infty, A)$ for the respective limits. Then $K_2(A)$ is the center of $\mathrm{St}(A)$ and $\mathrm{St}(A)\to E(A)$ is a universal central extension so that $K_2(A)=H_2(E(A))$. (If G is a group $H_2(G)$ stands for $H_2(G, \mathbf{Z})$ with trivial action on the coefficients.) One can define an analogue, $\mathrm{St}_n^*(A)$, of $\mathrm{St}_n(A)$ by taking a generator for each transvection in $\mathrm{GL}_n(A)$ and taking defining relations which mimic certain universal relations between transvections. This has the advantage that $\ker(\mathrm{St}_n^*(A)\to T_n(A))$ is automatically central in $\mathrm{St}_n^*(A)$. Moreover, if A is almost commutative and $n\geqslant 4$ it can be shown that the isomorphism $E_n(A)\to T_n(A)$ induces an isomorphism $\mathrm{St}_n(A)\to \mathrm{St}_n^*(A)$. So then $K_2(n, A)$ is also central. But I don't know if it is central for $n=3$, even for a polynomial ring in two variables over F_2. For $n=2$ counterexamples are known.

If I is a two-sided ideal in A, the *double* D is defined as the subring of $A\times A$ consisting of the (a, b) with $a-b\in I$. The relative Steinberg group $\mathrm{St}(A, I)$ is obtained as follows. (See Keune and Loday, References **[4]–[5]**.) The projection onto the first factor, $D\to A$, induces a homomorphism $\mathrm{St}(D)\to \mathrm{St}(A)$. Take its kernel. It contains commutators $[x_{12}((t, 0)), x_{21}((0, u))]$ for $t, u\in I$. Divide by the (central) subgroup generated by them. The result is $\mathrm{St}(A, I)$. (One can also define $\mathrm{St}(A, I)$ in St_n^*-style, without passing to the double.) Put $K_2(A, I)=\ker(\mathrm{St}(A, I)\to E(D))$. Recall that in higher algebraic K-theory there is a long exact sequence $\dots K_3(A/I)\to K_2(A, I)\to K_2(A)\to K_2(A/I)\to K_1(A, I)\dots$, which is the long exact homotopy sequence of the map $\mathrm{BGL}^+(A)\to \mathrm{BGL}^+(A/I)$. The above definitions are compatible with this.

2. Stability theorems. Conjecturally such theorems exist in a wider context but here we look only at $K_2(n, A)$ and $K_1(n, A, I)$. (Special case $A=I$.) So we ignore K_0. For special rings there are special results such as Dunwoody's theorem that, when A is euclidean, $K_2(2, A)\to K_2(r, A)$ is surjective for any $r\geqslant 3$. We now discuss the general results. The basic tool to prove them is Bass's stable range condition SR_n. We say that A satisfies SR_n if, for any unimodular row $\boldsymbol{a}=(a_1, \dots, a_n)$ of length n over A, there are $t_1, \dots, t_{n-1}\in A$ such that $(a_1+a_nt_1, \dots, a_{n-1}+a_nt_{n-1})$ is unimodular. Let me say that A satisfies SR_n^k (k-fold SR_n) if, given unimodular rows $\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(k)}$, each of length n, there are $t_1, \dots, t_{n-1}\in A$ which do the job for all k of them simultaneously. (There also exist stable range conditions for ideals. We ignore them here.) Recall that, for a right ideal J of A, a unimodular row $(a_1, \dots, a_n)$ is called *J-unimodular* if $a_1-1\in J$, $a_i\in J$ for $i>1$. Two such rows are *J-equivalent* if one can be obtained from the other by a finite sequence of steps in which a_i is replaced by a_i+a_jt with $j\neq i$ and $t\in A$ if $j>1$, $t\in J$ if $j=1$. For $n\geqslant 2$ consider the following conditions:

(A_n) A is finitely generated as a module over a central subring R, and this R has a noetherian maximal spectrum of dimension $\leqslant n-2$.

(B_n) A satisfies SR_n.

(C_n) A satisfies SR_n^2.

(D_n) For any right ideal J of A, all J-unimodular rows of length n are J-equivalent.

(D_n') Same with principal right ideals $J=aA$.

(E_n) For all two-sided ideals I of A, $K_1(r, A, I)\to K_1(A, I)$ is surjective for $r\geq n-1$ and injective for $r\geq n$.

(F_n) $K_2(r, A)\to K_2(A)$ is surjective for $r\geq n$ and injective for $r\geq n+1$.

Obviously, $(\mathrm{D}_n)\Rightarrow(\mathrm{D}_n')$ and $(\mathrm{C}_n)\Rightarrow(\mathrm{B}_n)$.

THEOREM (BASS, VASERŠTEĬN, DENNIS, SUSLIN, TULENBAYEV, VAN DER KALLEN). *For* $n\geq 2$, $(\mathrm{A}_n)\Rightarrow(\mathrm{B}_n)\Rightarrow[(\mathrm{C}_{n+1})\,\&\,(\mathrm{D}_n)]\Rightarrow \mathrm{E}_n$ *and* $[(\mathrm{C}_{n+1})\,\&\,(\mathrm{D}_n')]\Rightarrow(\mathrm{F}_n)$.

For $n\geq 3$, $(\mathrm{A}_n)\Rightarrow(\mathrm{C}_n)$.

So under the quite natural condition (A_n) we have the stability results (E_n), (F_n) and I have indicated possible technical intermediate results. Using $[(\mathrm{C}_{n+1})\,\&\,(\mathrm{D}_n')]\Rightarrow(\mathrm{F}_n)$, which is new, and the work of Bass, Milnor, Serre and Vaseršteĭn on the congruence subgroup problem for SL_2, I can now show the following. Let A be a subring of the algebraic closure of Q. Then if A is not contained in the ring of integers of its field of fractions or if this field is not totally imaginary, $K_2(2, A)\to K_2(A)$ is surjective and $K_2(3, A)\to K_2(A)$ is an isomorphism. This should be contrasted with a result of Dennis and Stein saying that $K_2(2, A)\to K_2(A)$ is not surjective when A is the ring of integers in $Q(\sqrt{d})$ where d is a squarefree rational integer, $d<-11$, d congruent to $-1 \bmod 8$ or to $-3 \bmod 9$. Let me finish this section by mentioning that Vaseršteĭn has solved the pre-stabilization problem for K_1 when A satisfies condition (A_n) of the theorem and $A/\mathrm{Rad}(A)$ has no zero divisors (Rad=Jacobson radical). That is, he gave generators for $\ker\big(K_1(n-1, A, I)\to K_1(A, I)\big)$.

3. Presentations for K_2. Presentations for K_2 have been obtained in two cases where stability is very strong, namely for commutative local rings and for relative K_2 of a radical ideal in a commutative ring.

(More precise results will follow.)

For a division ring D stability is also very strong but we do not know in general how to get explicit *generators* for $K_2(D)$. However, the pre-stabilization problem has been solved quite satisfactorily by Rehmann. He describes $K_2(D)$ as the kernel of a map $U_D\to[D^*, D^*]$. Here U_D may be viewed as $\mathrm{St}_1(D)/R$ where $\mathrm{St}_1(D)$ is some sort of rank 0 Steinberg group and R stands for $\ker\big(\mathrm{St}_1(D)\to\mathrm{St}(D)\big)$.

Let us restrict ourselves from now on to commutative rings. If R is semilocal we know by the above that $K_2(2, R)\to K_2(R)$ is surjective. In fact $K_2(R)$ is generated by the Dennis–Stein symbols $\langle a, b\rangle_{12}$. Here

$$\langle a, b\rangle_{12} = x_{21}(-b(1+ab)^{-1})x_{12}(a)x_{21}(b)x_{12}(-a(1+ab)^{-1})(h_{12}(1+ab))^{-1}$$

is defined for $a, b\in R$ when $1+ab\in R^*=\mathrm{GL}_1(R)$. One has $\langle ab, 1\rangle_{12}=1$, which might be used as a definition of $h_{12}(1+ab)$. Anyway, recall that $h_{ij}(t)$ is defined when $t \in R^*$, and that its image in $E(R)$ is a diagonal matrix. If $t, u \in R^*$, the Steinberg symbol $\{t, u\}_{12}$ is defined by $h_{12}(t)h_{12}(u)=\{t, u\}_{12}h_{12}(tu)$. (If both $1+ab$ and b are units, then $\langle a, b\rangle_{12}=\{1+ab, b\}_{12}$.) Let $US(R)$ denote the group of universal Steinberg symbols, which has a generator $\{t, u\}$ for each pair $t, u \in R^*$ and which has as defining relations (as an abelian group) $\{t, uv\}=\{t, u\}\{t, v\}$; $\{tu, v\}=\{t, v\}\{u, v\}$; $\{x, 1-x\}=1$. (As the relations have to make sense, one needs that x and $1-x$ are units.)

THEOREM (MATSUMOTO). *For a (commutative) field* F, $\{t, u\} \mapsto \{t, u\}_{12}$ *defines an isomorphism* $US(F) \to K_2(F)$.

I have shown that this result also holds for a ring satisfying SR_2^5, e.g. a local ring whose residue field contains at least 6 elements. But if one is not working with fields it is often better to use Dennis–Stein symbols.

Following Maazen and Stienstra let us define the group $D(R)$ as follows. Take a generator $\langle a, b\rangle$ for each pair $a, b \in R$ with $1+ab \in R^*$. Take defining relations (as an abelian group)

(D1) $\langle a, b\rangle\langle -b, -a\rangle = 1$.

(D2) $\langle a, b\rangle\langle a, c\rangle = \langle a, b+c+abc\rangle$.

(D3) $\langle a, bc\rangle = \langle ab, c\rangle\langle ac, b\rangle$.

(Stienstra now tells me I should use a different sign convention with $\langle a, b\rangle$ replaced by $\langle -a, b\rangle$.) For any commutative ring we have homomorphisms $US(R) \to D(R) \to K_2(R)$ sending $\{t, u\}$ to $\langle (t-1)u^{-1}, u\rangle$ and $\langle a, b\rangle$ to $\langle a, b\rangle_{12}$.

THEOREM. *If* R *is a commutative local ring,* $D(R) \to K_2(R)$ *is an isomorphism.*

The full proof of this theorem depends on work of Maazen–Stienstra, Dennis–Stein and myself. (Dennis and Stein in turn use the work of Matsumoto.) I have proved the same result for a commutative ring satisfying SR_2^3.

Now consider an ideal I with $I \subseteq \mathrm{Rad}\,(R)$. ($R$ is still commutative.) The group $D(R, I)$ is then defined just as $D(R)$, with the following modifications. Take generators $\langle a, b\rangle$ only if a or b is in I. Take relation (D3) only if a or b or c is in I. (And, as before, only consider relations that make sense.)

THEOREM. $D(R, I) \to K_2(R, I)$ *is an isomorphism.*

Here one sends $\langle a, b\rangle$ to $\langle (a, a), (0, b)\rangle_{12}$ or to $\langle (0, a), (b, b)\rangle_{12}$. (When both make sense they are equal.) If $R \to R/I$ splits, the theorem is due to Maazen and Stienstra. The present form was noted by Keune.

4. An example. Let R be a 1-dimensional commutative ring, finitely generated over a finite field. Let $A=R[T]$. We ask when $\mathrm{GL}_4(A)$ is finitely presented. Solution: Let φ_n denote substitution of T^n for T. Let $i \geq 0$. Put $V_n=K_i(\varphi_n)$.

As φ_n makes A into a free module of rank n over itself, we also have a transfer map $F_n\colon K_i(A)\to K_i(A)$, such that $F_nV_n=n(\mathrm{id})$. Further, if $\alpha\in NK_i(R)=\ker(K_i(A)\to K_i(R))$, there is a natural number M such that $F_n(\alpha)=0$ for $n\geqslant M$. (This is clear in $BQ(Nil)$ context.) From these properties of F_n, V_n it follows (cf. Farrell) that $NK_i(R)$ is either zero or not finitely generated. By Vaseršteĭn A satisfies SR_3, so that $K_i(4, A)\simeq K_i(A)$ for $i=1, 2$.

Now suppose R is regular. Then $K_i(A)\simeq K_i(R)$ is finitely generated by Quillen, so $K_1(4, A)$ and $K_2(4, A)$ are finitely generated. It follows, cf. Soulé and Rehmann, that $\mathrm{GL}_4(A)$ is finitely presented. (For smaller matrices such an argument would fail. Behr has shown that $\mathrm{SL}_3(\boldsymbol{F}_q[T])$ is not finitely presented, despite the fact that $SK_1(3, \boldsymbol{F}_q[T])=K_2(3, \boldsymbol{F}_q[T])=0$. Now if $q=2$ note that $\mathrm{SL}_3(\boldsymbol{F}_q[T])=\mathrm{GL}_3(\boldsymbol{F}_q[T])$.)

Conversely, suppose $\mathrm{GL}_4(A)$ is finitely presented. Then $K_1(4, A)$ is finitely generated, so $NK_1(R)$ is finitely generated and thus $NK_1(R)=0$. By Dennis there is a "noncanonical" homomorphism $p\colon H_2(\mathrm{GL}_4(A))\to K_2(A)$ whose composition with $H_2(E_4(A))\to H_2(\mathrm{GL}_4(A))$ is the usual map $H_2(E_4(A))\to K_2(A)$. So p is surjective. Now H_2 of a finitely presented group is finitely generated so $NK_2(R)$ must also be zero. By Vorst this can only happen if R is regular.

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