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Adjoint and Coadjoint Orbits of the Poincaré Group

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Abstract In this paper we give an effective method for finding a unique representative of each orbit of the adjoint and coadjoint action of the real affine orthogonal group on its Lie algebra. In both cases there are orbits which have a modulus that is different from the usual invariants for orthogonal groups. We find an unexplained bijection between adjoint and coadjoint orbits. As a special case, we classify the adjoint and coadjoint orbits of the Poincaré group.

Key words adjoint orbit · coadjoint orbit · cotype · type

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1. Introduction

Let $(\widetilde{V}, \widetilde{\gamma})$ be an n-dimensional real vector space with a nondegenerate inner product $\widetilde{\gamma}$. The set $\mathrm{O}(\widetilde{V}, \widetilde{\gamma})$ of real linear maps B of $\widetilde{V}$ into itself, which preserve $\widetilde{\gamma}$, that is, $\widetilde{\gamma}(Bv, Bw) = \widetilde{\gamma}(v, w)$ for every $v, w \in \widetilde{V}$, is a Lie group called the *orthogonal group*. Its Lie algebra $\mathrm{o}(\widetilde{V}, \widetilde{\gamma})$ consists of real linear maps ξ of $\widetilde{V}$ into itself such that $\widetilde{\gamma}(\xi v, w) + \widetilde{\gamma}(v, \xi w) = 0$ for every $v, w \in \widetilde{V}$. For $\xi, \eta \in \widetilde{V}$ the Lie bracket on $\mathrm{o}(\widetilde{V}, \widetilde{\gamma})$ is $[\xi, \eta] = \xi \circ \eta - \eta \circ \xi$, where $\circ$ is the composition of linear maps. The *affine orthogonal group* $\mathrm{Aff\ O}(\widetilde{V}, \widetilde{\gamma}) = \mathrm{O}(\widetilde{V}, \widetilde{\gamma}) \ltimes \widetilde{V}$ is the set of real affine orthogonal maps of $(\widetilde{V}, \widetilde{\gamma})$ into itself. More precisely, it is the set $\mathrm{O}(\widetilde{V}, \widetilde{\gamma}) \times \widetilde{V}$ with group multiplication $(B_1, v_1) \cdot (B_2, v_2) = (B_1 B_2, B_1 v_2 + v_1)$, which is the composition of affine linear maps. The affine orthogonal group is a Lie group. Its Lie algebra $\mathrm{affo}(\widetilde{V}, \widetilde{\gamma}) = \mathrm{o}(\widetilde{V}, \widetilde{\gamma}) \times \widetilde{V}$ has Lie bracket $[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1)$, where ξ_1, ξ_2 lie

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in $\mathrm{o}(\widetilde{V},\widetilde{\gamma})$. The *adjoint action* of the affine orthogonal group on its Lie algebra is defined by

$$\varphi : (\mathrm{O}(\widetilde{V},\widetilde{\gamma}) \ltimes \widetilde{V}) \times (\mathrm{o}(\widetilde{V},\widetilde{\gamma}) \times \widetilde{V}) \rightarrow \mathrm{o}(\widetilde{V},\widetilde{\gamma}) \times \widetilde{V} : \\ ((B,v),(\xi w)) \mapsto (B,v)\cdot(\xi,w)\cdot(B,v)^{-1}$$

where $\cdot$ is composition of affine linear maps. A straightforward calculation shows that $\varphi((B,v),(\xi,w)) = (B\xi B^{-1}, -B\xi B^{-1}v + Bw)$.

One of the goals of this paper is to classify the orbits of the adjoint action of the affine orthogonal group. In particular, we find a unique representative (= normal form) for each orbit. The basic technique leans heavily on the idea of an indecomposable type introduced by Burgoyne and Cushman [3] to find normal forms for the adjoint action of any real form of a nonexceptional Lie group. In this method the emphasis is not on subgroups and subvarieties, but rather on vector spaces with quadratic forms. (Indeed we learn little about an orbit as a variety. There is ample room for further work.)

Our aims are rather limited, but still we get results that seem to be new, despite a widespread belief that all is known on this topic. As explained in Section 2 below, our affine orthogonal group may be viewed as a subgroup of a slightly larger orthogonal group $\mathrm{O}(V,K)$. We find that the usual eigenvalue and Jordan invariants that classify the adjoint orbits of this ambient group $\mathrm{O}(V,K)$ do not suffice to distinguish the orbits of the affine orthogonal group. That is why we have to invent a modulus, which parametrizes families of adjoint orbits, each family being contained in a single orbit of $\mathrm{O}(V,K)$. In our classification of adjoint orbits we use the fact that we are working over the reals.

Next let us turn to the classification of coadjoint orbits, where one could deal with any base field of characteristic different from two. Recall that Rawnsley [5] has described how in principle one can classify the coadjoint orbits by reducing the problem to a similar problem for a subgroup known as the little subgroup. One should be careful though, because there is no canonical isomorphism between the little subgroup as an actual subgroup and your favorite incarnation of the isomorphism type of the little subgroup as a Lie group. This matters because affine orthogonal groups are less rigid than ordinary orthogonal groups. In particular, rescaling the vector part of an affine orthogonal group gives an automorphism that is not inner. Thus performing the actual classification, as opposed to giving an in principle classification, needs some care. We do the classification in the style of Burgoyne and Cushman [3], working with vector spaces instead of subgroups or subvarieties. Again we encounter an unfamiliar modulus. Surprisingly, once we have found representatives of coadjoint orbits, we see that there is a bijection between the chosen representatives for adjoint orbits and those employed for coadjoint orbits. This bijection preserves "dimension," "index," "modulus" and Jordan type. We have no geometric explanation for it.

We now give an overview of the contents of this paper. In Section 2 we show that the affine orthogonal group is a subgroup of a larger orthogonal group, which leaves an isotropic vector v° fixed. Throughout the remainder of the paper we look only at this isotropy group. In Section 3 we adapt the notion of an indecomposable type to the case at hand and show that there is a distinguished indecomposable type containing the vector v°. In Section 4 we classify these distinguished indecomposable types and complete the classification of the adjoint orbits of the affine orthogonal group. In Section 5 we apply the above theory to find normal forms for the adjoint orbits of the Poincaré group. In Section 6 we classify the coadjoint orbits of the affine orthogonal group and in Section 7 we specialize this to the coadjoint orbits of the Poincaré group.

2. Affine Orthogonal Group

In this section we show that the affine orthogonal group can be realized as an isotropy subgroup of a larger orthogonal group.

Let $\widetilde{\gamma}$ be a nondegenerate inner product on a real n-dimensional vector space $\widetilde{V}$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\widetilde{V}$ such that the matrix of $\widetilde{\gamma}$ with respect to this basis is $G = \mathrm{diag}(-I_m, I_p)$, where I_r is the $r \times r$ identity matrix. Let $\mathrm{O}(\widetilde{V}, G)$ be the set of all linear maps B of $\widetilde{V}$ into itself which preserve $\widetilde{\gamma}$, that is, $\widetilde{\gamma}(Bv, Bw) = \widetilde{\gamma}(v, w)$ for every $v, w \in \widetilde{V}$. Then $\mathrm{O}(\widetilde{V}, G)$ is a Lie group which is isomorphic to $\mathrm{O}(m, p)$. On $V = \mathbb{R} \times \widetilde{V} \times \mathbb{R}$ consider the inner product γ defined by $\gamma((x, v, y), (x', v', y')) = \widetilde{\gamma}(v, v') + x'y + xy'$. With respect to the basis $\mathfrak{e} = \{e_0, e_1, \ldots, e_n, e_{n+1}\}$ of V the matrix of γ is *standard*, that is, $K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & G & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Note that e_{n+1} is a *K-isotropic* vector of (V, K), that is, $K(e_{n+1}, e_{n+1}) = 0$. Let $\mathrm{O}(V, K)$ be the set of all real linear maps A of V into itself which preserve γ, that is, $\gamma(A(x, v, y), A(x', v', y')) = \gamma((x, v, y), (x', v', y'))$.

Now consider the isotropy subgroup

$$\mathrm{O}(V, K)_{e_{n+1}} = \{A \in \mathrm{O}(V, K) \;\big|\; Ae_{n+1} = e_{n+1}\}$$

of $\mathrm{O}(V, K)$. To give a more explicit description of $\mathrm{O}(V, K)_{e_{n+1}}$ let A be an invertible real linear map of V into itself such that $Ae_{n+1} = e_{n+1}$. Suppose that the matrix of A with respect to the basis $\mathfrak{e}$ is $\begin{pmatrix} a & b^T & c \\ d & B & e \\ f & g^T & h \end{pmatrix}$. Then $A = \begin{pmatrix} a & b^T & 0 \\ d & B & 0 \\ f & g^T & 1 \end{pmatrix}$, because A leaves the vector e_{n+1} fixed. Now $A \in \mathrm{O}(V, K)$ if and only if $K = A^T K A$, that is,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ d & B & 0 \\ -\frac{1}{2} d^T G d & -d^T G B & 1 \end{pmatrix}, \tag{1}$$

where $B^T G B = G$ and $d \in \mathbb{R}^n$. Thus $A \in \mathrm{O}(V, K)_{e_{n+1}}$ if and only if (1) holds. The group $\mathrm{O}(V, K)_{e_{n+1}}$ is isomorphic to the *affine orthogonal* group $\mathrm{AffO}(V, K)$, which is the semidirect product $\ltimes$ of $\mathrm{O}(\mathbb{R}^n, G)$ with $\mathbb{R}^n$, that is,

$$\mathrm{O}(\mathbb{R}^n, G) \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} 1 & 0 \\ d & B \end{pmatrix} \in \mathrm{Gl}(n+1, \mathbb{R}) | B^T G B = G, \; d \in \mathbb{R}^n \right\}.$$

Explicitly, the isomorphism is given by

$$\mathrm{O}(V, K)_{e_{n+1}} \to \mathrm{O}(\mathbb{R}^{n+1}, G) \ltimes \mathbb{R}^n : \begin{pmatrix} 1 & 0 & 0 \\ d & B & 0 \\ -\frac{1}{2} d^T G d & -d^T G B & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ d & B \end{pmatrix}.$$

We determine the Lie algebra $\mathrm{o}(V,K)_{e_{n+1}}$ of $\mathrm{O}(V,K)_{e_{n+1}}$ as follows. Let $v \in \mathbb{R}^n$ and $X \in \mathrm{o}(\widetilde{V}, G)$, that is, $X^T G + GX = 0$. Then

$$t \mapsto \begin{pmatrix} 1 & 0 & 0 \\ tv & \exp tX & 0 \\ -\frac{1}{2}(tv)^T G(tv) & -(tv)^T G \exp tX & 1 \end{pmatrix} = Y_t$$

is a curve in $\mathrm{O}(V,K)_{e_{n+1}}$ which passes through the identity element at $t = 0$. Consequently, $\frac{d}{dt}\Big|_{t=0} Y_t = \left(\begin{smallmatrix} 0 & 0 & 0 \\ v & X & 0 \\ 0 & -v^T G & 0 \end{smallmatrix}\right)$ is an element of $\mathrm{o}(V,K)_{e_{n+1}}$. The Lie bracket $[\,,\,]$ on $\mathrm{o}(V,K)_{e_{n+1}}$ is given by

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ v_1 & X_1 & 0 \\ 0 & -v_1{}^T G & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ v_2 & X_2 & 0 \\ 0 & -v_2^T G & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 \\ X_1 v_2 - X_2 v_1 & [X_1, X_2] & 0 \\ 0 & -(X_1 v_2 - X_2 v_1)^T G & 0 \end{pmatrix},$$

where $[X_1, X_2]$ is the Lie bracket in $\mathrm{o}(\widetilde{V}, G)$.

3. Classification of Adjoint Orbits

To fix notation. Let v° be a nonzero *isotropic* vector in the real inner product space (V, γ). Let $\mathrm{o}(V,\gamma)_{v^\circ}$ be the Lie algebra of the affine orthogonal group

$$\mathrm{O}(V,\gamma)_{v^\circ} = \{A \in \mathrm{Gl}(V) | Av^\circ = v^\circ \text{ and } A^*\gamma = \gamma\}.$$

Then $Y \in \mathrm{o}(V,\gamma)_{v^\circ}$ if and only if

$$Yv^\circ = 0 \text{ and } \gamma(Yv, w) + \gamma(v, Yw) = 0, \text{ for all } v, w \in V.$$

We begin our classification of the adjoint orbits of the affine orthogonal group $\mathrm{O}(V,\gamma)_{v^\circ}$ on its Lie algebra $\mathrm{o}(V,\gamma)_{v^\circ}$ by defining the notions of indecomposable type and indecomposable distinguished type. First we define the notion of a pair. Let W be a γ-nondegenerate real vector space. Our vector spaces are always finite dimensional. If $Y \in \mathrm{o}(W,\gamma)$ then $(Y, W; \gamma)$ is a *pair*.[1] We say that the pairs $(Y, W; \gamma)$ and $(Y', W'; \gamma')$ are *equivalent* if there is a bijective real linear map $P: W \to W'$ such that $PY = Y'P$ and $P^*\gamma' = \gamma$, that is, $\gamma'(Pv, Pw) = \gamma(v, w)$ for every $v, w \in W$. Clearly being equivalent is an equivalence relation on the collection of pairs. An equivalence class of pairs is a *type*, which we denote by Δ. Given a type Δ with representative $(Y, W; \gamma)$ we define the *dimension*, denoted $\dim \Delta$, of Δ by $\dim W$ and the *index*, denoted $\operatorname{ind} \Delta$, of Δ by the number of negative eigenvalues of the Gram matrix $(\gamma(v_i, v_j))$, where $\{v_1, \ldots, v_{\dim W}\}$ is a basis of W. It is straightforward to check that neither of these notions depends on the choice of representative of Δ or on the choice of basis. Let $Y = S + N$ be the Jordan decomposition of Y into a semisimple linear map S and a commuting nilpotent linear map N. Because S and N are polynomials in Y with real coefficients and $Yv^\circ = 0$, it follows that $Sv^\circ = Nv^\circ = 0$. So $S, N \in \mathrm{o}(W,\gamma)_{v^\circ}$. Let h be the unique nonnegative integer such that $N^h W \neq \{0\}$ but $N^{h+1} W = 0$. We call h the *height* of the type Δ and we denote it by $ht(\Delta)$. It is evident that $ht(\Delta)$ does not depend of the choice of representative of Δ. We say that a type Δ with representative $(Y, W; \gamma)$ is *uniform* if $NW = \ker N^h | W$. Let $(Y, W; \gamma)$ represent the type Δ.

[1] Our concept of pair is the same as that of [3].

Suppose that $W = W_1 + W_2$, where W_i are proper, Y-invariant subspaces, which are γ-nondegenerate and γ orthogonal. Then Δ is the *sum* of two types Δ_i, which are represented by $(Y|W_i, W_i; \gamma|W_i)$. Δ is *indecomposable* if it cannot be written as the sum of two types. From [3, prop. 3, p.343] it follows that an indecomposable type is uniform. So far the vector v° has not played any role. Therefore the classification of indecomposable types is given by results in [3].

We now define the notion of a triple, where the vector v° plays an essential role. $(Y, W, v^\circ; \gamma)$ is a *triple* if and only if the vector v° is nonzero and γ-isotropic and for the linear map Y in the pair $(Y, W; \gamma)$ we have $Yv^\circ = 0$. We say that the triple $(Y, W, v^\circ; \gamma)$ is a *nilpotent triple* if Y is nilpotent. Two triples $(Y, W, v^\circ; \gamma)$ and $(Y', W', (v^\circ)'; \gamma')$ are *equivalent* if there is a bijective real linear map $P : W \to W'$ such that $Y'P = PY$, $P^*\gamma' = \gamma$ and $Pv^\circ = (v^\circ)'$. Clearly being equivalent is an equivalence relation on the collection of triples. We call an equivalence class of triples a *distinguished type*, which we denote by $\underline{\Delta}$. Let $(Y, W, v^\circ; \gamma)$ represent the distinguished type $\underline{\Delta}$. If Y is nilpotent, then $\underline{\Delta}$ is a *nilpotent distinguished type*. Suppose that $W = W_1 \oplus W_2$, where W_i are proper, Y-invariant, γ-orthogonal, γ-degenerate subspaces and $v^\circ \in W_1$. Then $(Y|W_1, W_1; \gamma|W_1)$ is a triple whose distinguished type we write $\underline{\Delta}_1$. Moreover, let the pair $(Y|W_2, W_2; \gamma|W_2)$ represent the type Δ_2. In this situation we say that the distinguished type $\underline{\Delta}$ is the *sum* of the distinguished type $\underline{\Delta}_1$ and the type Δ_2 and we write $\underline{\Delta} = \underline{\Delta}_1 + \Delta_2$. If $\underline{\Delta}$ cannot be written as the sum of a distinguished type and a type, then we say that $\underline{\Delta}$ is an *indecomposable* distinguished type. In other words, $(Y, W, v^\circ; \gamma)$ represents an indecomposable distinguished type if there is no proper, γ-nondegenerate, Y-invariant subspace of W which contains v°. To simplify notation from now on we usually drop the inner product γ in pairs and triples.

The first goal of this paper is to prove

THEOREM 1. *Every distinguished type is a sum of an indecomposable nilpotent distinguished type and a sum of indecomposable types. This decomposition is unique up to a reordering of the summands.*

The proof of the theorem will require an understanding of indecomposable nilpotent distinguished types. Recall the indecomposable types have already been classified in [3]. The theorem solves the conjugacy class problem for the Lie algebra $\mathrm{o}(v, \gamma)_{v^\circ}$. Indeed distinguished types represented by triples of the form $(Y, V, v^\circ; \gamma)$ correspond one to one with orbits of the adjoint action on $\mathrm{o}(v, \gamma)_{v^\circ}$.[2]

Before beginning the proof of Theorem 1, we need some additional concepts. Let $\underline{\Delta}$ be a distinguished type with representative (Y, W, v°) .We say that $\underline{\Delta}$ has *distinguished height h*, if h is the largest positive integer for which there is a vector $w \in W$ such that $Y^h w = v^\circ$. We denote the distinguished height of $\underline{\Delta}$ by $\mathrm{dht}(\underline{\Delta})$. Because the definition of distinguished height does not involve the inner product γ and $Yv^\circ = 0$, there is a largest Jordan block of the linear map Y which contains the vector v°. Moreover, it is of size $h + 1$. Let

$$\mu(\underline{\Delta}) = \{\gamma(w, v^\circ) \in \mathbb{R} |\ \text{for all } w \in W \text{ such that } Y^h w = v^\circ\}.$$

[2] Let us warn readers, who prefer to work with arbitrary perfect fields of characteristic different from two, that Theorem 1 fails in such generality because the results of [3] do not carry over.

We call $\mu(\underline{\Delta})$ the set of *parameters* of the distinguished type $\underline{\Delta}$. Below we will show that this set is a singleton.

We prove

LEMMA 2. *Suppose that* $\underline{\Delta} = \underline{\Delta}' + \Delta$. *Then* $\mathrm{dht}(\underline{\Delta}) = \mathrm{dht}(\underline{\Delta}')$ *and* $\mu(\underline{\Delta}) = \mu(\underline{\Delta}')$.

Proof. Suppose that (Y, W, v°) is a triple which represents the distinguished type $\underline{\Delta}$ and that $W = W_1 \oplus W_2$, where W_i are proper, Y-invariant, γ-orthogonal, γ-nondegenerate subspaces of W with $v^\circ \in W_1$. Say the triple $(Y|W_1, W_1, v^\circ)$ represents a distinguished type $\underline{\Delta}'$ and the pair $(Y|W_2, W_2)$ represents the type Δ. Suppose that $\mathrm{dht}(\underline{\Delta}') = h'$. Then there is a vector $w' \in W_1$ such that $Y^{h'} w' = v^\circ$. Consequently, $\mathrm{dht}(\underline{\Delta}) \geq h'$. Since $\mathrm{dht}(\underline{\Delta}) = h$, there is a vector $w \in W$ such that $Y^h w = v^\circ$. But $W = W_1 \oplus W_2$. So we may write $w = w_1 + w_2$ where $w_i \in W_i$. Since W_i are Y-invariant, we have $v^\circ = Y^h w_1 + Y^h w_2$ where $Y^h w_i \in W_i$. By construction $v^\circ \in W_1$. Therefore $Y^h w_1 = v^\circ$. Consequently $h \leq \mathrm{dht}(\underline{\Delta}') = h'$. So $h = h'$. Note that $\dim \underline{\Delta} > \dim \underline{\Delta}'$.

Since $W_1 \subseteq W$, it follows from the definition of the set of parameters that $\mu(\underline{\Delta}') \subseteq \mu(\underline{\Delta})$. Suppose that there is a vector $w \in W$ with $Y^h w = v^\circ$ such that $\gamma(w, v^\circ) \notin \mu(\underline{\Delta}')$. Write $w = w_1 + w_2$ where $w_i \in W_i$. Then by the argument in the preceding paragraph we find that $Y^h w_1 = v^\circ$. Since W_2 is γ-orthogonal to W_1 and $v^\circ \in W_1$, we obtain

$$\gamma(w, v^\circ) = \gamma(w_1, v^\circ) + \gamma(w_2, v^\circ) = \gamma(w_1, v^\circ).$$

But $\gamma(w_1, v^\circ) \in \mu(\underline{\Delta}')$ by definition. This is a contradiction. Hence $\mu(\underline{\Delta}') = \mu(\underline{\Delta})$. □

LEMMA 3. *We may write* $\underline{\Delta} = \underline{\Delta}' + \Delta$ *where the distinguished type* $\underline{\Delta}'$ *is indecomposable and nilpotent.*

Proof. If the distinguished type $\underline{\Delta}'$ is not indecomposable, we find another distinguished type $\underline{\Delta}''$ of the same distinguished height and parameters and a type Δ' such that $\underline{\Delta}' = \underline{\Delta}'' + \Delta'$, where $\dim \Delta' > 0$. Because $\dim \underline{\Delta}' > \dim \underline{\Delta}''$after a finite number of repetitions, we obtain a distinguished type $\underline{\widetilde{\Delta}}$ which we can no longer write as a sum of a distinguished type and a type, namely, $\underline{\Delta} = \underline{\widetilde{\Delta}} + \widetilde{\Delta}$. In other words, $\underline{\widetilde{\Delta}}$ is an indecomposable distinguished type. By Lemma 2 it has the same distinguished height and parameters as the distinguished type $\underline{\Delta}$.

We now show that the indecomposable distinguished type $\underline{\widetilde{\Delta}}$, represented by $(Y|W, W, v^\circ)$, is nilpotent. Let W_0 be the generalized eigenspace of $Y|W$ corresponding to the eigenvalue 0. Then W_0 is Y-invariant, γ-nondegenerate and contains v°. On W_0 the linear map Y is nilpotent. From the fact that the distinguished type $\underline{\widetilde{\Delta}}$ is indecomposable, it follows that the triple $(Y|W_0, W_0, v^\circ;\ \gamma)$ equals the triple $(Y|W, W, v^\circ; \gamma)$. Hence the indecomposable distinguished type $\underline{\widetilde{\Delta}}$ is nilpotent. □

4. Indecomposable Distinguished Types

In this section we classify indecomposable distinguished types. We start by giving a rough description of the possible indecomposable distinguished types, which we then refine to a classification.

Let $\underline{\Delta}$ be a distinguished type. There are two cases:

1. the set of parameters $\mu(\underline{\Delta})$ contains a nonzero parameter; or
2. $\mu(\underline{\Delta}) = \{0\}$.

Case 1. Suppose that the triple (Y, W, v°) represents the distinguished type $\underline{\Delta}$, which we assume has distinguished height h. Using Lemma 3 write $\underline{\Delta} = \underline{\Delta}' + \Delta$, where $\underline{\Delta}'$ is an indecomposable distinguished type of distinguished height h represented by $(Y|W_1, W_1, v^\circ)$ with W_1 a γ-nondegenerate, Y-invariant subspace of W which contains v°. Choose $w \in W_1$ so that $Y^h w = v^\circ$ and $\gamma(w, v^\circ) = \mu \neq 0$.[3] Look at the subspace

$$\widetilde{W} = \operatorname{span}\{w,\ Yw,\ \dots,\ Y^h w\}$$

of W. Clearly $v^\circ \in \widetilde{W}$. On $\widetilde{W}$ consider the $(h+1) \times (h+1)$ Gram matrix $G = \big(\gamma(Y^i w, Y^j w)\big) = \big(\pm\gamma(w, Y^{i+j} w)\big)$, since $Y \in \mathrm{o}(W, \gamma)_{v^\circ}$. Because $Y^{h+1} w = Y v^\circ = 0$, we have $Y^{h+1}|\widetilde{W} = 0$. Therefore, all the entries of G below the antidiagonal are 0. On the other hand, because

$$\gamma(Y^i w, Y^{h-i} w) = \pm\gamma(w, Y^h w) = \pm\mu \neq 0,$$

all the entries of G on the antidiagonal are nonzero. Hence $\det G \neq 0$, that is, $\widetilde{W}$ is γ-nondegenerate. As $\underline{\Delta}'$ was assumed to be indecomposable, it follows that $W_1 = \widetilde{W}$. Note that $(Y|\widetilde{W}, \widetilde{W}, v^\circ)$ has one Jordan block and therefore $\underline{\Delta}'$ is uniform. This completes case 1.

Case 2. Suppose that the triple (Y, W, v°) represents the distinguished type $\underline{\Delta}$, which we assume has distinguished height h. Using Lemma 3 write $\underline{\Delta} = \underline{\Delta}' + \Delta$, where $\underline{\Delta}'$ is a nilpotent indecomposable distinguished type of distinguished height h represented by $(Y|W_1, W_1, v^\circ)$ with W_1 a γ-nondegenerate, Y-invariant subspace of W which contains v°. Consider the pair $(Y|W_1, W_1)$ and the type $\widetilde{\Delta}$ which it represents. From the results of [3] we may write $\widetilde{\Delta} = \Delta_1 + \cdots + \Delta_r$, where Δ_j are indecomposable types uniform of height h_j, sorted so that $h_1 \leq h_2 \leq \cdots \leq h_r$. Suppose that $(Y|W_j, W_j)$ represents Δ_j. Then v° is a sum of its components in the W_j, but some of those components may be zero. Let $\widehat{W} = W_k$ where k is the smallest index such that v° has a nonzero component $\widehat{v^\circ}$ in $\widehat{W}$. Consider the type $(Y|\widehat{W}, \widehat{W})$. Then $Y|\widehat{W}$ annihilates $\widehat{v^\circ}$ and the height of $(Y|\widehat{W}, \widehat{W})$ equals the distinguished height h of $\underline{\Delta}'$. Choose $z \in \widehat{W}$ such that $\gamma(z, v^\circ) = \gamma(z, \widehat{v^\circ}) \neq 0$. This is possible since $\widehat{W}$ is γ-nondegenerate. Choose $w \in W_1$ so that $Y^h w = v^\circ$. Consider the Y-invariant subspace

$$\widetilde{W} = \operatorname{span}\{w,\ Yw,\ \dots\ Y^h w;\ z,\ Yz, \dots,\ Y^h z\}.$$

[3] This implies that h is even. Suppose not. Then

$$\gamma(w, Y^h w) = (-1)^h\, \gamma(Y^h w, w) = -\gamma(w, Y^h w),$$

since γ is symmetric. Hence $\gamma(w, Y^h w) = 0$, which is a contradiction.

Let $n = h+1$. Note that $Y^{h+1}|\tilde{W} = 0$ and $\gamma(z, Y^h w) \neq 0$ by definition of z and w. Moreover $\gamma(w, Y^h w) = 0$ since $\mu(\underline{\Delta}) = \{0\}$ by hypothesis. Look at the $2n \times 2n$ Gram matrix

$$G = \left(\begin{array}{c|c} g_{i,j} & g_{i,j+n} \\ \hline g_{i+n,j} & g_{i+n,j+n} \end{array}\right) = \left(\begin{array}{c|c} \gamma(Y^{i-1}w, Y^{j-1}w) & \gamma(Y^{i-1}w, Y^{j-1}z) \\ \hline \gamma(Y^{i-1}z, Y^{j-1}w) & \gamma(Y^{i-1}z, Y^{j-1}z) \end{array}\right).$$

The entries of G satisfy the following conditions: i) $g_{i,j} = g_{n+i,j} = g_{i,n+j} = g_{n+i,n+j} = 0$, when $i+j \geq n+2$ and $1 \leq i,j \leq n$; ii) $g_{i,j+n} = g_{i+n,j} \neq 0$, where $i+j = n+1$; iii) $g_{i,j} = 0$, where $i+j = n+1$. Thus G has its nonzero entries on or above the antidiagonal of each $n \times n$ block except the upper left hand one, where even the antidiagonal elements are zero. Thus the matrix G has the form

$$\left(\begin{array}{cc|cc} * & 0 & * & + \\ & & & \\ 0 & 0 & + & 0 \\ \hline * & + & * & * \\ & & & \\ + & 0 & * & 0 \end{array}\right),$$

where $+$ denotes a nonzero entry. Expanding $\det G$ by minors of the $h+1^{\text{st}}$ column, one sees that $\det G$ is a nonzero number times the $[h+2, h+1]$ minor. Expanding this minor by its last column gives a nonzero number times a matrix with the same form as the original G but with one fewer row and column. Clearly when G is a 2×2 , we have $\det G \neq 0$. By induction we have

LEMMA 4. $\det G = \pm \prod_{k=1}^{2n} g_{k,2n-k+1} \neq 0$.

Thus $\tilde{W}$ is a $2h+2$-dimensional, Y-invariant, γ nondegenerate subspace of W_1, which contains the vector v°. Since $\underline{\Delta}'$ is indecomposable, the triple $(Y|\tilde{W}, \tilde{W}, v^\circ)$ represents the distinguished type $\underline{\Delta}'$. Note that $\underline{\Delta}'$ is made up of two Jordan blocks of size $h+1$ and hence is uniform. This completes case 2 of the rough description of indecomposable distinguished types.

We now classify indecomposable distinguished types.

PROPOSITION 5. *Let $\underline{\Delta}$ be an indecomposable distinguished type of distinguished height h, which is represented by the triple (Y, W, v°). Then exactly one of the following alternatives holds.*

1. *h is even, $h > 0$, and there is a basis*

$$\{w,\ Yw, \ldots, Y^{h/2-1}w;\ \varepsilon\, Y^h w, -\varepsilon\, Y^{h-1}w, \ldots, (-1)^{h/2-1}\varepsilon Y^{h/2+1}w;\ Y^{h/2}w\}, \quad (2)$$

where the Gram matrix of γ is $\begin{pmatrix} 0 & I_{h/2} & 0 \\ I_{h/2} & 0 & 0 \\ 0 & 0 & (-1)^{h/2}\varepsilon \end{pmatrix}$ and $v^\circ = \mu Y^h w$ with $\mu > 0$. We call μ a modulus. Here $\varepsilon^2 = 1$. We use the notation $\underline{\Delta}_h^\varepsilon(0),\ \mu$.

2. *h is odd and there is a basis*

$$\{Y^h z, -Y^{h-1}z, \ldots, (-1)^h z\,; w,\ Yw,\ \ldots,\ Y^h w\}, \quad (3)$$

where the Gram matrix of γ is $\begin{pmatrix} 0 & I_{h+1} \\ I_{h+1} & 0 \end{pmatrix}$ and $v^\circ = Y^h w$. We use the notation $\underline{\Delta}_h(0,0)$.

3. *h is even and there is a basis*

$$\{Y^h z,\ -Y^{h-1}z,\dots,(-1)^h z\,;w,\ Yw,\ \dots,Y^h w\},\tag{4}$$

where the Gram matrix of γ *is* $\begin{pmatrix} 0 & I_{h+1} \\ I_{h+1} & 0 \end{pmatrix}$ *and* $v^\circ = Y^h w$. *We use the notation* $\underline{\Delta_h^+(0) + \Delta_h^-(0)}$.

Proof. Using our rough classification of distinguished indecomposable types, let us prove the proposition.

Suppose that we are in case 1 of the rough classification. Then $\underline{\Delta}$ is represented by the triple (Y, W, v°) where $W = \mathrm{span}\{w,\ Yw,\ \dots,Y^h w\}$ and $\gamma(w, Y^h w) \neq 0$. Hence h is even and $h > 0$ because v° is isotropic, while $\gamma(w, Y^h w) \neq 0$. Since $\underline{\Delta}$ is uniform we may form $\overline{W} = W/YW$. Clearly, $\dim \overline{W} = 1$. On $\overline{W}$ the inner product γ induces a symmetric bilinear form $\overline{\gamma}$ defined by $\overline{\gamma}(\overline{v}, \overline{v}') = \gamma(v, Y^h v')$. Since $\gamma(w, Y^h w) \neq 0$, the vector $\overline{w}$ is nonzero and forms a basis of $\overline{Y}$. Rescaling, we may assume that $\overline{\gamma}(\overline{w}, \overline{w}) = \varepsilon$, where $\varepsilon^2 = 1$. By [3, prop. 2, p.343] any uniform type is determined by its height and its $(\overline{W}, \overline{\gamma})$, so we may choose a vector $w \in W$ which generates the basis (2) of case 1 of the proposition, γ-adapted in the sense that its Gram matrix is as indicated in the proposition. Indeed such a γ-adapted basis describes a type that has the required height and $(\overline{W}, \overline{\gamma})$. In terms of this basis there is a unique nonzero number μ such that $v^\circ = \mu Y^h w$. Replacing w with $-w$, if necessary, we can assume that $\mu > 0$. We call μ a modulus. We compute that

$$\gamma(\mu w, v^\circ) = \overline{\gamma}(\mu\overline{w}, \mu\overline{w}) = \mu^2\,\varepsilon,$$

which shows that $\mu(\underline{\Delta}) = \{\mu^2\varepsilon\}$. Thus $\mu(\underline{\Delta})$ determines μ and ε. So $\underline{\Delta}$ is a distinguished indecomposable type made up of one Jordan block. Moreover, we have $\dim \underline{\Delta} = h+1$, $\mathrm{ind}\,\underline{\Delta} = \begin{cases} h/2, & \text{if } (-1)^{h/2}\varepsilon = 1 \\ h/2+1, & \text{if } (-1)^{h/2}\varepsilon = -1 \end{cases}$ and $\underline{\Delta}$ has distinguished height h and a unique modulus $\mu > 0$. The type of (Y, W) is denoted $\Delta_h^\varepsilon(0)$ in [3].

Now suppose that we are in case 2 of the rough classification. Then the distinguished type $\underline{\Delta}$ of distinguished height h is represented by the triple (Y, W, v°) with

$$W = \mathrm{span}\{w,\ Yw,\dots,Y^h w,\ z,\ Yz,\ \dots,Y^h z\},$$

and $v^\circ = Y^h w$. Moreover, $\gamma(w, v^\circ) = 0$ and $\gamma(z, v^\circ) \neq 0$. There are two subcases.

Suppose that h is odd. Since $\underline{\Delta}$ is uniform, we may form $\overline{W} = W/YW$. On $\overline{W}$ the inner product γ induces a skew symmetric bilinear form $\overline{\gamma}$ defined by $\overline{\gamma}(\overline{v}, \overline{v}') = \gamma(v, Y^h v')$. Clearly, $\overline{W} = \mathrm{span}\{\overline{w}, \overline{z}\}$ and from $\overline{\gamma}(\overline{w}, \overline{z}) \neq 0$ it follows that $\overline{W}$ is $\overline{\gamma}$ nondegenerate. Up to isomorphism there is only one nondegenerate skew symmetric bilinear form of dimension two, and it is indecomposable. So $\overline{W}$ is $\overline{\gamma}$ indecomposable. Using [3, prop. 2, p.343] again we may choose vectors $w, z \in W$ which generate the γ-adapted basis (3) of case 2 of the proposition. Then $\overline{\gamma}$ has matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with respect to the basis $\{\overline{w}, \overline{z}\}$. We now need to show that we can choose the γ-adapted basis so that $v^\circ = Y^h w$. We know that $v^\circ = \alpha Y^h w + \beta Y^h z$ is a nonzero vector in $\ker Y|W$. If $\alpha \neq 0$, let $\binom{w'}{z'} = \begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} \binom{w}{z}$; while if $\alpha = 0$ let $\binom{w'}{z'} = \begin{pmatrix} 0 & \beta \\ -1/\beta & 0 \end{pmatrix} \binom{w}{z}$. We rewrite the definition as $\binom{w'}{z'} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \binom{w}{z}$, where $ad - bd = 1$. We now show that w' and z' generate the γ-adapted basis $\{Y^h z', -Y^{h-1} z', \ldots, (-1)^h z'; w', Yw', \ldots, Y^h w'\}$ of W. This follows because for every j between 0 and h we have

$$\gamma(Y^i w', Y^j w') = \gamma(Y^i z', Y^j z') = 0$$

and

$$\begin{aligned} \gamma(Y^j w', (-1)^j Y^{h-j} z') &= (-1)^j \gamma(Y^j(aw + bz), Y^{h-j}(cw + dz)) \\ &= \gamma(aw + bz, Y^h(cw + dz)) \\ &= ad\,\overline{\gamma}(\overline{w}, \overline{w}) + bd\,\overline{\gamma}(\overline{z}, \overline{z}) + (ad - bc)\overline{\gamma}(\overline{w}, \overline{z}) \\ &= \overline{\gamma}(\overline{w}, \overline{z}) = 1. \end{aligned}$$

By construction $v^\circ = Y^h w'$. Summarizing, we have shown that $\underline{\Delta}$ is a distinguished indecomposable type made up of two Jordan blocks. Also $\dim \underline{\Delta} = 2(h+1)$, $\operatorname{ind} \underline{\Delta} = h + 1$ and $\underline{\Delta}$ has distinguished height h, which is odd. The type of (Y, W) is denoted $\Delta_h(0, 0)$ in [3].

Suppose that h is even. Since $\underline{\Delta}$ is uniform, we may form $\overline{W} = W/YW$. On $\overline{W}$ the inner product γ induces a symmetric bilinear form $\overline{\gamma}$ defined by $\overline{\gamma}(\overline{v}, \overline{v}') = \gamma(v, Y^h v')$. Since $\gamma(z, Y^h w) \neq 0$ by hypothesis, we see that $\gamma(\overline{z}, \overline{w}) \neq 0$ and $\overline{W} = \operatorname{span}\{\overline{z}, \overline{w}\}$. Therefore the reduced type $(\overline{Y}, \overline{W}; \overline{\gamma})$ is *not* indecomposable. Since $\gamma(w, Y^h w) = 0$, the vector $\overline{w}$ is a nonzero and $\overline{\gamma}$-isotropic. Let $\overline{y} = \frac{1}{\gamma(\overline{z}, \overline{w})}\left(\overline{z} - \frac{\gamma(\overline{z}, \overline{z})}{2\gamma(\overline{z}, \overline{w})}\overline{w}\right)$. Then $\overline{y}$ is a $\overline{\gamma}$-isotropic vector in $\overline{W}$ and $\overline{\gamma}(\overline{y}, \overline{w}) = 1$. Thus the matrix of $\overline{\gamma}$ with respect to the basis $\{\overline{y}, \overline{w}\}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Using [3, prop. 2, p.343] we may choose vectors $\widetilde{w}, \widetilde{z} \in W$ which generate the γ-adapted basis (4) of case 3 of the proposition. We now need to show that we can choose this basis so that $v^\circ = Y^h \widetilde{w}$. Since $v^\circ \in \ker Y|W$, we see that $v^\circ \in \operatorname{span}\{Y^h \widetilde{w}, Y^h \widetilde{z}\}$. Now write $v^\circ = Y^h(\alpha \widetilde{w} + \beta \widetilde{z})$. As $\gamma(\alpha \widetilde{w} + \beta \widetilde{z}, v^\circ) = 2\alpha\beta \in \mu(\underline{\Delta}) = \{0\}$, we must have $\alpha = 0$ or $\beta = 0$. If $v^\circ = \alpha Y^h \widetilde{w}$, where $\alpha \neq 0$, then put $z' = \alpha^{-1} \widetilde{z}$, $w' = \alpha \widetilde{w}$. If $v^\circ = \beta Y^h \widetilde{z}$ with $\beta \neq 0$ then put $z' = \beta^{-1} \widetilde{w}$, $w' = \beta \widetilde{z}$. In either case $v^\circ = Y^h w'$ and

$$\{Y^h z', -Y^{h-1} z', \ldots, (-1)^h z'; w', Yw', \ldots, Y^h w'\}$$

is a basis of W with respect to which the matrix of γ is $\begin{pmatrix} 0 & I_{h+1} \\ I_{h+1} & 0 \end{pmatrix}$. Note $\underline{\Delta}$ is a distinguished indecomposable type made up of two Jordan blocks. Also $\dim \underline{\Delta} = 2(h+1)$ with $\operatorname{ind} \underline{\Delta} = h + 1$ and $\underline{\Delta}$ has distinguished height h, which is even. The type of (Y, W) is decomposable and is denoted $\Delta_h^+(0) + \Delta_h^-(0)$ in [3].

One may look at the above computation as exploiting the fact that there is an action of $\mathrm{O}(\overline{W}, \overline{\gamma})$ on $\ker Y|W$. In the last two cases the action has only one orbit of nonzero isotropic vectors, while in the first case there are moduli. The action can be understood in terms of the Jacobson Morozov theorem.

The three cases are obviously exclusive. Note that one can distinguish them by $\operatorname{dht}(\underline{\Delta})$ and $\mu(\underline{\Delta})$. This proves Proposition 5. □

Proof of Theorem 1. Let $\underline{\Delta}$ be a distinguished type. By Lemma 3 we may write $\underline{\Delta} = \underline{\widetilde{\Delta}} + \Delta$ where the distinguished type $\underline{\widetilde{\Delta}}$ is indecomposable and nilpotent. By the main result of [3, theorem, p.343] applied to Δ, we can write

$$\underline{\Delta} = \underline{\widetilde{\Delta}} + \Delta_1 + \cdots + \Delta_r, \tag{5}$$

where Δ_i for $1 \le i \le r$ are indecomposable types. By Lemma 2, $\underline{\widetilde{\Delta}}$ is of the same distinguished height and parameters as $\underline{\Delta}$. Suppose that $\underline{\Delta}$ has another such decomposition, namely

$$\underline{\Delta} = \underline{\widetilde{\Delta}}' + \Delta_1' + \cdots + \Delta_s', \tag{6}$$

where $\underline{\widetilde{\Delta}}'$ is an indecomposable distinguished type and Δ_j' for $1 \le j \le s$ are indecomposable types. By Lemma 2 the distinguished height, say h, of $\underline{\widetilde{\Delta}}$ and $\underline{\widetilde{\Delta}}'$ are the same. Say that $\underline{\widetilde{\Delta}}$ and $\underline{\widetilde{\Delta}}'$ are represented by the triples (Y, W, v°) and $(Y', W', (v^\circ)')$. Suppose that h is odd. Then the linear map $P : W \to W'$ for which $PY^i w = (Y')^i w'$ and $PY^i z = (Y')^i z'$ where $0 \le i \le h$ and w, z and w', z' are vectors given in the basis (3) of case 2 of Proposition 5 is an equivalence between the triples (Y, W, v°) and $(Y', W', (v^\circ)')$. Next suppose that h is even and that (Y, W, v°) and $(Y', W', (v^\circ)')$ have one Jordan chain. Since by Lemma 2 the parameters of $\underline{\widetilde{\Delta}}$ and $\underline{\widetilde{\Delta}}'$ are the same, using the basis (2) of case 1 of Proposition 5 we can again construct an equivalence between $\underline{\widetilde{\Delta}}$ and $\underline{\widetilde{\Delta}}'$. We can also handle the case when h is even and (Y, W, v°) and $(Y', W', (v^\circ)')$ have two Jordan chains. Thus in every case (Y, W, v°) and $(Y', W', (v^\circ)')$ are equivalent, that is, $\underline{\widetilde{\Delta}} = \underline{\widetilde{\Delta}}'$.

Now we need only show that $r = s$ and $\Delta_i = \Delta'_i$. But this follows from the the main result of [3, theorem, p.343], because $\Delta_1 + \cdots \Delta_r$ and $\Delta_1' + \cdots \Delta_s'$ are sums of indecomposable types, while $\underline{\widetilde{\Delta}} = \underline{\widetilde{\Delta}}'$ implies that the underlying types of $\underline{\widetilde{\Delta}}$ and $\underline{\widetilde{\Delta}}'$ are equal. This proves Theorem 1. □

5. Adjoint Orbits of the Poincaré Group

In this section we use the above theory to determine the orbits of the adjoint action of the *Poincaré* group on its Lie algebra.

Let $G = \mathrm{diag}(-1, -1, -1, 1)$ be the matrix of a Lorentz inner product on $\mathbb{R}^4$ with respect to the standard basis $\{e_1, \ldots, e_4\}$. The Poincare group is the affine Lorentz group, which is the semidirect product $\mathrm{O}(3,1) \ltimes \mathbb{R}^4$ of the Lorentz group $\mathrm{O}(3,1) = \mathrm{O}(\mathbb{R}^4, G)$ with the abelian group $\mathbb{R}^4$. In Section 2 we have shown that the Poincaré group is the isotropy group $\mathrm{O}(\mathbb{R}^6, K)_{e_5}$ of the orthogonal group $\mathrm{O}(\mathbb{R}^6, K)$, where the matrix of the inner product K with respect to the basis $\{e_0, e_1, \ldots, e_4, e_5\}$ of $\mathbb{R}^6$ is standard. The Lie algebra of the Poincaré group is isomorphic to the Lie algebra $\mathrm{o}(\mathbb{R}^6, K)_{e_5}$ of $\mathrm{O}(\mathbb{R}^6, K)_{e_5}$. All the conjugacy classes in $\mathrm{o}(\mathbb{R}^6, K)_{e_5}$ are given in Table 3 below.

First we list all the possible $\mathrm{o}(\mathbb{R}^6, K)_{e_5}$-indecomposable distinguished types, meaning indecomposable distinguished types that may occur as summand of some $(Y, \mathbb{R}^6, e_5; K)$.

Note we express v° using the basis given in Proposition 5.

We now show that all the possible indecomposable distinguished types are listed in Table 1. The possible eigenvalue combinations are $0\,0$; 0; and $0 + 0$. Here, for

Table 1 Possible $o(\mathbb{R}^6, K)_{e_5}$-indecomposable distinguished types

	Type (modulus $\alpha > 0$)	Dim	Index	v°
1.	$\underline{\Delta_4^-(0)}$, $\alpha > 0$	5	3	$\alpha Y^4 w$
2.	$\underline{\Delta_4^+(0)}$, $\alpha > 0$	5	2	$\alpha Y^4 w$
3.	$\underline{\Delta_1(0,0)}$	4	2	Yw
4.	$\underline{\Delta_2^+(0)}$, $\alpha > 0$	3	2	$\alpha Y^2 w$
5.	$\underline{\Delta_2^-(0)}$, $\alpha > 0$	3	1	$\alpha Y^2 w$
6.	$\underline{\Delta_0^+(0) + \Delta_0^-(0)}$	2	1	w

instance, $0 + 0$ stands for a decomposable two dimensional $(\overline{Y}, \overline{W}; \overline{\gamma})$ with eigenvalue zero for each summand. The corresponding heights and signs are 1; $4^\pm$, $2^\pm$; and 0. So Table 1 lists all the possibilities.

Next we list the possible $o(\mathbb{R}^6, K)_{e_5}$-indecomposable types, see [3, Table 2, p.349]. That is, we look for types that occur as proper summand of some $(Y, \mathbb{R}^6; K)$. We do not claim they all actually occur in the setting of Theorem 1.

Note in Table 2 we have used the notation $\Delta_m(\zeta, CQ) = \Delta_m(\zeta, -\zeta, \overline{\zeta}, -\overline{\zeta})$, $\zeta \neq \pm\overline{\zeta}$, $\Delta_m(\zeta, RP) = \Delta_m(\zeta, -\zeta)$, $\zeta = \overline{\zeta} \neq 0$, $\Delta_m(\zeta, IP) = \Delta_m(\zeta, -\zeta)$, $\zeta = -\overline{\zeta} \neq 0$, where ζ is the complex eigenvalue of Y with $(Y, W; K)$ a representative of the $o(\mathbb{R}^6, K)$-indecomposable type. For instance, $\Delta_m(\zeta, -\zeta, \overline{\zeta}, -\overline{\zeta})$ has height m and four eigenvalues on $\overline{W}$.

We now show that all the possible $o(\mathbb{R}^6, K)_{e_5}$-indecomposable types are listed in Table 2. For each eigenvalue combination we have the following possibilities for the heights and the signs, because the dimension is at most five.

eigenvalues	CQ	IP	RP	0	$0\,0$
height and sign	0	$0^\pm$, $1^\pm$	0, 1	$0^\pm$, $2^\pm$, $4^\pm$	1.

This gives a total of 14 cases, two of which are covered by case 5. Thus Table 2 is complete.

Next we combine a given distinguished type in Table 1 with a sum of $o(\mathbb{R}^6, K)_{e_5}$-indecomposable types from Table 2 so that their dimensions add up to 6 and their indices add up to 4.

The list below of dimension–index pairs shows that all the $O(\mathbb{R}^6, K)_{e_5}$-conjugacy classes in $o(\mathbb{R}^6, K)_{e_5}$ are given in Table 3.

Table 2 Possible $o(\mathbb{R}^6, K)_{e_5}$-indecomposable types

	Type	Dim	Index
1.	$\Delta_4^-(0)$	5	3
2.	$\Delta_4^+(0)$	5	2
3.	$\Delta_0(\zeta, CQ)$	4	2
4.	$\Delta_1(\zeta, RP)$	4	2
5.	$\Delta_1^\varepsilon(\zeta, IP)$	4	2
6.	$\Delta_1(0,0)$	4	2
7.	$\Delta_2^+(0)$	3	2
8.	$\Delta_2^-(0)$	3	1
9.	$\Delta_0^-(\zeta, IP)$	2	2
10.	$\Delta_0(\zeta, RP)$	2	1
11.	$\Delta_0^+(\zeta, IP)$	2	0
12.	$\Delta_0^-(0)$	1	1
13.	$\Delta_0^+(0)$	1	0

Below we show how to find explicit normal forms from the decomposition into an indecomposable distinguished $o(\mathbb{R}^6, K)_{e_5}$-type and a sum of indecomposable $o(\mathbb{R}^6, K)_{e_5}$-types given in Table 3. We do this for one case just to give the idea.

EXAMPLE 6. $\underline{\Delta_4^-(0)}$, $\alpha + \Delta_0^-(0)$.

Write $\mathbb{R}^6 = V_1 \oplus V_2$, where V_1 and V_2 are Y-invariant, K-orthogonal, $o(\mathbb{R}^6, K)_{e_5}$-indecomposable subspaces where $(V_1, Y|V_1) \in \Delta_4^-(0)$, $e_5 \in V_1$, and $(V_2, Y|V_2) \in \Delta_0^-(0)$. Now $Y = N$ is nilpotent on V_1 and V_2. Choose a basis

$$\{v_1, Nv_1, -N^4 v_1, N^3 v_1; N^2 v_1\}$$

of V_1 as in case 1 of Proposition 5. Note that $v^\circ = \alpha N^4 v_1$ with $\alpha > 0$. Also there is a vector v_2 in V_2 such that $K(v_2, v_2) = -1$. With respect to the basis

$$\{e_0, \ldots, e_5\} = \{-\alpha^{-1} v_1, \tfrac{1}{2} N v_1 - N^3 v_1, N^2 v_1, v_2, \tfrac{1}{2} N v_1 + N^3 v_1; \alpha N^4 v_1\}$$

the matrix of K is standard while the matrix of $Y \in o(\mathbb{R}^6, K)_{e_5}$ is

$$\left(\begin{array}{c|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -\alpha^{-1} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha^{-1} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & -\alpha^{-1} & 0 & 0 & \alpha^{-1} & 0 \end{array}\right),$$

which is the desired normal form.

Table 3 Conjugacy classes in $o(\mathbb{R}^6, K)_{e_5}$

	Indecomposable distinguished type (modulus $\alpha > 0$)	Sum of $o(\mathbb{R}^6, K)_{e_5}$ indecomposable types	Dim	Index
1.	$\underline{\Delta_4^-(0)}$, α		5	3
a.		$+\Delta_0^-(0)$	1	1
2.	$\underline{\Delta_1(0,0)}$		4	2
a.		$+\Delta_0^-(\zeta, IP)$	2	2
b.		$+\Delta_0^-(0) + \Delta_0^-(0)$	2	2
3.	$\underline{\Delta_2^+(0)}$, α		3	2
a.		$+\Delta_2^+(0)$	3	2
b.		$+\Delta_0^-(\zeta, IP) + \Delta_0^+(0)$	3	2
c.		$+\Delta_0(\zeta, RP) + \Delta_0^-(0)$	3	2
d.		$+\Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^+(0)$	3	2
4.	$\underline{\Delta_2^-(0)}$, α		3	1
a.		$+\Delta_0^-(\zeta, IP) + \Delta_0^-(0)$	3	3
b.		$+\Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^-(0)$	3	3
5.	$\underline{\Delta_0^+(0) + \Delta_0^-(0)}$		2	1
a.		$+\Delta_2^+(0) + \Delta_0^-(0)$	4	3
b.		$+\Delta_0^-(\zeta, IP) + \Delta_0(\zeta, RP)$	4	3
c.		$+\Delta_0^-(\zeta, IP) + \Delta_0^-(0) + \Delta_0^+(0)$	4	3
d.		$+\Delta_0(\zeta, RP) + \Delta_0^-(0) + \Delta_0^-(0)$	4	3
e.		$+\Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^+(0)$	4	3

	Dimension–index pair	Dimension–index pairs in sum of indecomposable types
1.	(5,3)	(1,1)
2.	(4,2)	(2,2), (1,1) + (1,1)
3.	(3,2)	(3,2), (2,2) + (1,0), (2,1) + (1,1), (1,1) + (1,1) + (1,0)
4.	(3,1)	(2,2) + (1,1), (1,1) + (1,1) + (1,1)
5.	(2,1)	(3,2) + (1,1), (2,1) + (1,1) + (1,1), (2,2) + (1,1) + (1,0), (2,2) + (2,1), (1,1) + (1,1) + (1,1) + (1,0).

6. Classification of Coadjoint Orbits

Our next aim is to determine a representative of each orbit of the coadjoint action

$$\mathrm{O}(\mathbb{R}^6, K)_{e_5} \times \mathrm{o}(\mathbb{R}^6, K)^*_{e_5} \to \mathrm{o}(\mathbb{R}^6, K)^*_{e_5} : (P, Y^*) \mapsto Y^* \circ Ad_{P^{-1}}$$

of the Poincaré group $\mathrm{O}(\mathbb{R}^6, K)_{e_5}$ on the dual $\mathrm{o}(\mathbb{R}^6, K)^*_{e_5}$ of its Lie algebra. More generally, we classify the coadjoint orbits of an affine orthogonal group. As before, it is essential to our method that the affine orthogonal group is viewed as an isotropy subgroup. Instead of types we will now employ cotypes.

As always, the pair (V, γ) is a finite dimensional real vector space with a nondegenerate inner product γ. When K is the Gram matrix of γ with respect to some basis, we often write K for γ. For a vector v in V let v^* be the linear function on V given by $w \mapsto \gamma(v, w)$. A *tuple* $(V, Y, v; \gamma)$ is a pair (V, γ), a real linear map $Y \in \mathrm{o}(V, \gamma)$ and a vector $v \in V$. On the collection of all tuples we say that the tuple $(V, Y, v; \gamma)$ is *equivalent* to the tuple $(V', Y', v'; \gamma')$ if and only if there is a bijective real linear map $P : V \to V'$ such that (i) $P^*\gamma' = \gamma$, (ii) $Pv = v'$, and (iii) there is a vector $w \in V$ such that $Y' = P(Y + L_{w,v})P^{-1}$, where $L_{w,v} = w \otimes v^* - v \otimes w^*$.

FACT 1. $L_{w,v} \in \mathrm{o}(V, \gamma)$.

FACT 2. *If* $P \in \mathrm{O}(V, \gamma)$, *then* $PL_{w,v}P^{-1} = L_{Pw,Pv}$.

Being equivalent is an equivalence relation on the collection of tuples. An equivalence class is a *cotype*, which is denoted by ∇. If $(V, Y, v; \gamma)$ is a representative of ∇, then define the *dimension* of ∇ to be $\dim V$ and denote it by $\dim \nabla$. Clearly, the notion of dimension is well defined. A cotype is *affine* if it has a representative $(V, Y, v; \gamma)$, where v is a nonzero, γ-isotropic vector.

Suppose that we are in the situation of Section 2, where $V = \mathbb{R} \times \widetilde{V} \times \mathbb{R}$ is a real vector space with nondegenerate inner product γ defined by

$$\gamma((x, \widetilde{v}, y), (x', \widetilde{v}', y')) = \widetilde{\gamma}(\widetilde{v}, \widetilde{v}') + x'y + y'x,$$

where $\widetilde{\gamma}$ is a nondegenerate inner product on $\widetilde{V}$. Suppose that with respect to the standard basis $\mathfrak{e} = \{e_0, e_1, \ldots, e_n, e_{n+1}\}$ of V the matrix of γ is $K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & G & 0 \\ 1 & 0 & 0 \end{pmatrix}$, where G is the matrix of $\widetilde{\gamma}$ with respect to the basis $\widetilde{\mathfrak{e}} = \{e_1, \ldots e_n\}$ of $\widetilde{V}$.

The following proposition explains the relevance of affine cotypes. See also Proposition 13 below.

PROPOSITION 9. *For $Y \in \mathrm{o}(V,\gamma)$ let ℓ_Y be the linear function on $\mathrm{o}(V,\gamma)$ which maps Z to $\mathrm{tr}\, YZ$. The map $(V,Y,e_{n+1};K) \mapsto \ell_Y|\mathrm{o}(V,K)_{e_{n+1}}$ induces a bijection between affine cotypes on (V,K) and coadjoint orbits of $\mathrm{O}(V,K)_{e_{n+1}}$ on the dual $\mathrm{o}(V,K)^*_{e_{n+1}}$ of its Lie algebra.*

Proof. The argument is a series of observations.

Suppose that the tuples $(V,Y,e_{n+1};K)$ and $(V,Y',e_{n+1};K)$ are equivalent. Then there is a real linear map $P \in \mathrm{O}(V,K)_{e_{n+1}}$ and a vector $w \in V$ such that $Y' = P(Y + L_{w,e_{n+1}})P^{-1}$.

OBSERVATION 1. *The matrix of $L_{w,e_{n+1}}$* with respect to the standard basis $\mathfrak{e}$ of (V,K) is $\begin{pmatrix} w_0 & 0 & 0 \\ \widetilde{w} & 0 & 0 \\ 0 & -\widetilde{w}^T G & -w_0 \end{pmatrix}$, where $w = w_0 e_0 + \widetilde{w} + w_{n+1}e_{n+1} \in V$.

Proof. We compute

$$\begin{aligned} L_{w,e_{n+1}}(e_0) &= (e_{n+1}^T K e_0)w - (w^T K e_0)e_{n+1} = w - w_{n+1}e_{n+1} \\ &= w_0 e_0 + \widetilde{w}; \\ L_{w,e_{n+1}}(e_i) &= (e_{n+1}^T K e_i)w - (w^T K e_i)e_{n+1} = -(\widetilde{w}^T G e_i)e_{n+1}; \\ L_{w,e_{n+1}}(e_{n+1}) &= (e_{n+1}^T K e_{n+1})w - (w^T K e_{n+1})e_{n+1} = -w_0 e_{n+1}. \end{aligned}$$

□

OBSERVATION 2. *For $P \in \mathrm{O}(V,K)$ and $Y \in \mathrm{o}(V,K)$ we have*

$$\ell_{PYP^{-1}} = \mathrm{Ad}^T_{P^{-1}} \ell_Y := \ell_Y \circ \mathrm{Ad}_{P^{-1}}.$$

Proof. Let $Z \in \mathrm{o}(V,K)$. Then

$$\begin{aligned} \ell_{PYP^{-1}}(Z) &= \mathrm{tr}\left(P(YP^{-1}Z)\right) = \mathrm{tr}\left((YP^{-1}Z)P\right) \\ &= \mathrm{tr}\left(Y(P^{-1}ZP)\right) = \ell_Y(\mathrm{Ad}_{P^{-1}}Z) \\ &= (\mathrm{Ad}^T_{P^{-1}}\ell_Y)Z. \end{aligned}$$

□

OBSERVATION 3. *We have*

$$\{L_{v,e_{n+1}} \mid v \in V\} = \{Y \in \mathrm{o}(V,K) \mid \ell_Y \text{ vanishes on } \mathrm{o}(V,K)_{e_{n+1}}\}.$$

Proof. With respect to the standard basis $\mathfrak{e}$ of (V,K), the matrix of $L_{v,e_{n+1}}$ is $\begin{pmatrix} v_0 & 0 & 0 \\ \widetilde{v} & 0 & 0 \\ 0 & -\widetilde{v}^T G & -v_0 \end{pmatrix}$, where $v = v_0 e_0 + \widetilde{v} + v_{n+1}e_{n+1}$ and the matrix of $Z \in \mathrm{o}(V,K)_{e_{n+1}}$ is $\begin{pmatrix} 0 & 0 & 0 \\ \widetilde{y} & X & 0 \\ 0 & -\widetilde{y}^T G & 0 \end{pmatrix}$, where $\widetilde{y} \in \widetilde{V}$ and $X \in \mathrm{o}(\widetilde{V},G)$. Then

$$\begin{aligned} \ell_{L_{v,e_{n+1}}}(Z) &= \mathrm{tr}\left[\begin{pmatrix} v_0 & 0 & 0 \\ \widetilde{v} & 0 & 0 \\ 0 & -\widetilde{v}^T G & -v_0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ \widetilde{y} & X & 0 \\ 0 & -\widetilde{y}^T G & 0 \end{pmatrix}\right] \\ &= \mathrm{tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\widetilde{v}^T G\widetilde{y} & v_0\widetilde{y}^T G - \widetilde{v}^T GX & 0 \end{pmatrix} = 0. \end{aligned}$$

So $\mathrm{o}(V,K)_{e_{n+1}} \subseteq \ker \ell_{L_{v,e_{n+1}}}$. Now

$$\dim \mathrm{o}(V,K) - \dim \mathrm{o}(V,K)_{e_{n+1}} = n+1 = \dim\{L_{v,e_{n+1}} \in \mathrm{o}(V,K)\,, v \in V\}.$$

As $(Y,Z) \mapsto \ell_Y Z$ is nondegenerate on $\mathrm{o}(V,K)$, the result follows. □

Now we are in position to prove the proposition. For $Y \in \mathrm{o}(V,K)$, $w \in V$ and $Z \in \mathrm{o}(V,K)_{e_{n+1}}$, we calculate

$$\begin{aligned} \ell_{P(Y+L_{w,e_{n+1}})P^{-1}}(Z) &= \ell_{PYP^{-1}}(Z) + \ell_{PL_{w,e_{n+1}}P^{-1}}(Z) \\ &= (\mathrm{Ad}^T_{P^{-1}}\ell_Y)Z + \ell_{L_{Pw,e_{n+1}}}(Z), \text{since } P \in \mathrm{O}(V,K)_{e_{n+1}} \\ &= (\mathrm{Ad}^T_{P^{-1}}\ell_Y)Z. \end{aligned}$$

In other words,

$$\ell_{P(Y+L_{w,e_{n+1}})P^{-1}} | \mathrm{o}(V,K)_{e_{n+1}} = (\mathrm{Ad}^T_{P^{-1}} Y) | \mathrm{o}(V,K)_{e_{n+1}}.$$

Thus the affine cotype represented by $(V,Y,e_{n+1};K)$ corresponds to a unique coadjoint orbit of $\mathrm{O}(V,K)_{e_{n+1}}$ on the dual of its Lie algebra $\mathrm{o}(V,K)^*_{e_{n+1}}$. □

Suppose that we are given the affine cotype ∇ with representative $(\widehat{V},\widehat{Y},\widehat{v};\widehat{\gamma})$. We wish to associate a Gram matrix K to it. For this, recall that the distinguished type, represented by $(0,\widehat{V},\widehat{v};\widehat{\gamma})$, has a representative of the form $(0,V,e_{n+1};K)$, where $V = \mathbb{R}\times\widetilde{V}\times\mathbb{R}$ and $K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & G & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We may replace the representative of the cotype ∇ with one of the form $(V,Y,e_{n+1};K)$, where the matrix of Y with respect to the standard basis $\mathfrak{e}$ is $\begin{pmatrix} y_0 & -\widetilde{v}^* & 0 \\ \widetilde{y} & \widetilde{Y} & \widetilde{v} \\ 0 & -\widetilde{y}^* & -y_0 \end{pmatrix}$. Here $y_0 \in \mathbb{R}$, $\widetilde{v},\widetilde{y} \in \widetilde{V}$ and $\widetilde{Y} \in \mathrm{o}(\widetilde{V},G)$. We say that the cotype ∇_ℓ, represented by $(\widetilde{V},\widetilde{Y},\widetilde{v};G)$, is the *little cotype* of ∇.[4]

LEMMA 10. *The little cotype ∇_ℓ does not depend on the choice of representative of the affine cotype ∇.*

Proof. Up to isomorphism $(\widetilde{V},G)$ is determined by ∇, so there is no need to vary G or K. Let $(V,Y,e_{n+1};K)$ be a representative of the affine cotype ∇. Suppose that $(V,Y',e_{n+1};K)$ is another representative. Then there is a $P \in \mathrm{O}(V,K)_{e_{n+1}}$ and a vector $w \in V$ such that

$$Y' = P(Y + L_{w,e_{n+1}})P^{-1}. \tag{7}$$

We now calculate the right hand side of (7) explicitly. With respect to the standard basis $\mathfrak{e}$ of (V,K), we have $P = \begin{pmatrix} 1 & 0 & 0 \\ \widetilde{u} & A & 0 \\ -\frac{1}{2}\widetilde{u}^T G\widetilde{u} & -\widetilde{u}^T GA & 1 \end{pmatrix}$, where $\widetilde{u} \in \widetilde{V}$ and $A \in \mathrm{O}(\widetilde{V},G)$. Therefore $P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -A^{-1}\widetilde{u} & A^{-1} & 0 \\ -\frac{1}{2}\widetilde{u}^T G\widetilde{u} & \widetilde{u}^T G & 1 \end{pmatrix}$. Moreover, $Y = \begin{pmatrix} y_0 & -\widetilde{v}^T G & 0 \\ \widetilde{y} & \widetilde{Y} & \widetilde{v} \\ 0 & -\widetilde{y}^T G & -y_0 \end{pmatrix}$, where $y_0 \in \mathbb{R}$, $\widetilde{v},\widetilde{y} \in \widetilde{V}$ and $\widetilde{Y} \in \mathrm{o}(\widetilde{V},G)$, and $L_{w,e_{n+1}} = \begin{pmatrix} w_0 & 0 & 0 \\ \widetilde{w} & 0 & 0 \\ 0 & -\widetilde{w}^T G & -w_0 \end{pmatrix}$, where $w = w_0 e_0 + \widetilde{w} + w_{n+1}e_{n+1}$. So

[4] The cotype ∇_ℓ is called the little cotype because we are imitating the little subgroup approach of Wigner [7] to the representation theory of the Poincaré group.

$$
\begin{aligned}
Y' &= P(Y + L_{w,e_{n+1}})P^{-1} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ \widetilde{u} & A & 0 \\ -\frac{1}{2}\widetilde{u}^T G\widetilde{u} & -\widetilde{u}^T GA & 1 \end{pmatrix} \begin{pmatrix} * & -\widetilde{v}^T G & 0 \\ * & \widetilde{Y} & \widetilde{v} \\ 0 & * & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -A^{-1}\widetilde{u} & A^{-1} & 0 \\ -\frac{1}{2}\widetilde{u}^T G\widetilde{u} & \widetilde{u}^T G & 1 \end{pmatrix} \\
&= \begin{pmatrix} * & -\widetilde{v}^T G & 0 \\ * & -\widetilde{u}\otimes\widetilde{v}^T G + A\widetilde{Y} & A\widetilde{v} \\ * & * & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -A^{-1}\widetilde{u} & A^{-1} & 0 \\ -\frac{1}{2}\widetilde{u}^T G\widetilde{u} & \widetilde{u}^T G & 1 \end{pmatrix} \\
&= \begin{pmatrix} b_0 & -(\widetilde{v})'^T G & 0 \\ \widetilde{b} & \widetilde{Y}' & \widetilde{v}' \\ 0 & -\widetilde{b}^T G & -b_0 \end{pmatrix},
\end{aligned}
$$

where $b_0 \in \mathbb{R}$, $\widetilde{b} \in \widetilde{V}$,

$$
\begin{aligned}
\widetilde{Y}' &= A\widetilde{Y}A^{-1} - \widetilde{u}\otimes\widetilde{v}^T GA^{-1} + A\widetilde{v}\otimes\widetilde{u}^T G \\
&= A\widetilde{Y}A^{-1} - \widetilde{u}\otimes(A\widetilde{v})^* + A\widetilde{v}\otimes\widetilde{u}^* \\
&= A\widetilde{Y}A^{-1} + L_{-\widetilde{u},A\widetilde{v}},
\end{aligned}
$$

and $\widetilde{v}' = A\widetilde{v}$. Thus the little cotype ∇_ℓ, as computed from $(V, Y', e_{n+1}; K)$, is represented by the tuple $(\widetilde{V}, \widetilde{Y}', \widetilde{v}'; G)$, which does not depend on the vector w. Since $\widetilde{Y}' = A(\widetilde{Y} + L_{-A^{-1}\widetilde{u},\widetilde{v}})A^{-1}$ and $\widetilde{v}' = A\widetilde{v}$, the tuple $(\widetilde{V}, \widetilde{Y}', \widetilde{v}'; G)$ is equivalent to the tuple $(\widetilde{V}, \widetilde{Y}, \widetilde{v}; G)$. But this tuple depends only on the representative $(V, Y, e_{n+1}; K)$ and *not* the representative $(V, Y', e_{n+1}; K)$ of the cotype ∇. So the little cotype ∇_ℓ does not depend on the choice of representative of the affine cotype ∇. □

LEMMA 11. *Let ∇ be an affine cotype. Then ∇ is uniquely determined by its little cotype ∇_ℓ.*

Proof. Suppose that the affine cotypes ∇ and ∇', represented by the tuples $(V, Y, v; \gamma)$ and $(V, Y', v; \gamma)$, both have the little cotype ∇_ℓ. Say $Y = \begin{pmatrix} u_0 & -\widetilde{w}^T G & 0 \\ \widetilde{u} & \check{Y} & \widetilde{w} \\ 0 & -\widetilde{u}^T G & 0 \end{pmatrix}$ with $u_0 \in \mathbb{R}$, $\widetilde{u}, \widetilde{w} \in \widetilde{V}$ and $\check{Y} \in \mathrm{o}(\widetilde{V}, G)$ and $Y' = \begin{pmatrix} u_0' & -(\widetilde{w}')^T G & 0 \\ \widetilde{u}' & \check{Y}' & \widetilde{w}' \\ 0 & -(\widetilde{u}')^T G & 0 \end{pmatrix}$ with $u_0' \in \mathbb{R}$, $\widetilde{u}', \widetilde{w}' \in \widetilde{V}$ and $\check{Y}' \in \mathrm{o}(\widetilde{V}, G)$. Thus ∇_ℓ has both a representative $(\widetilde{V}, \check{Y}, \widetilde{w}; G)$ and a representative $(\widetilde{V}, \check{Y}', \widetilde{w}'; G)$. Hence $(\widetilde{V}, \check{Y}, \widetilde{w}; G)$ is equivalent to $(\widetilde{V}, \check{Y}', \widetilde{w}'; G)$. In other words, there is a $\widehat{A} \in \mathrm{O}(\widetilde{V}, G)$ and a vector $\widetilde{u} \in \widetilde{V}$ such that $\widehat{A}\widetilde{w} = \widetilde{w}'$ and

$$
\check{Y}' = \widehat{A}(\check{Y} + L_{\widetilde{u},\widetilde{w}})\widehat{A}^{-1}.
$$

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{A} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $A \in \mathrm{O}(V, K)_{e_{n+1}}$ and

$$
\begin{aligned}
A(Y + L_{\widetilde{u},\widetilde{w}})A^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{A} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 & -\widetilde{w}^T G & 0 \\ \widetilde{u} & \check{Y} + L_{\widetilde{u},\widetilde{w}} & \widetilde{w} \\ 0 & -\widetilde{u}^T G & -u_0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{A}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} u_0 & -\widetilde{w}^T G\widehat{A}^{-1} & 0 \\ \widehat{A}\widetilde{u} & \widehat{A}(\check{Y} + L_{\widetilde{u},\widetilde{w}})\widehat{A}^{-1} & \widehat{A}\widetilde{w} \\ 0 & -\widetilde{u}^T G\widehat{A}^{-1} & -u_0 \end{pmatrix} \\
&= Y' + L_{Au-u',e_{n+1}},
\end{aligned}
$$

where $u = u_0 e_0 + \widetilde{u}$ (hence $Au = u_0 + \widehat{A}\,\widetilde{u}$) and $u' = u_0' e_0 + \widetilde{u}'$. Consequently, the tuples $(V, Y, e_{n+1}; K)$ and $(V, Y', e_{n+1}; K)$ are equivalent. So the cotypes ∇ and ∇' are equal. □

Remark 12. Given a cotype ∇_ℓ it is very easy to construct a cotype ∇ having ∇_ℓ as little cotype. Indeed if $(\widetilde{V}, \widetilde{Y}, \widetilde{v}; G)$ represents ∇_ℓ, one forms V, K in the usual way and takes a representative of the form $(V, Y, e_{n+1}; K)$, where the matrix of Y with respect to the standard basis $\mathfrak{e}$ is $\begin{pmatrix} 0 & -\widetilde{v}^* & 0 \\ 0 & \widetilde{Y} & \widetilde{v} \\ 0 & 0 & 0 \end{pmatrix}$.

The following proposition follows immediately from the above.

PROPOSITION 13. *There is a bijection between little cotypes and coadjoint orbits.*

Let ∇ be a cotype represented by the tuple $(V, Y, v; \gamma)$. Suppose that $V = V_1 \oplus V_2$, where V_i are Y-invariant, γ-nondegenerate and γ-orthogonal subspaces such that $V_2 \neq \{0\}$ and $v \in V_1$. Then we say that ∇ is a *sum* of the cotype $\widetilde{\nabla}$, represented by the tuple $(V_1, Y|V_1, v; \gamma|V_1)$, and a *type* Δ, represented by $(Y|V_2, V_2; \gamma|V_2)$. We write $\nabla = \widetilde{\nabla} + \Delta$. If $V_1 = \{0\}$, then $v = 0$ and $\widetilde{\nabla}$ is the *zero cotype*, represented by the tuple $(\{0\}, 0, 0; 0)$ and denoted by 0. We say that a cotype is *indecomposable* if it cannot be written as the sum of a cotype and a type. A nonzero cotype ∇, represented by the tuple $(V, Y, v; \gamma)$ is decomposable if there is a proper, Y-invariant subspace of V, which contains the vector v and on which γ is nondegenerate. Conversely, if ∇ is decomposable, then there is a representative $(V, Y, v; \gamma)$ so that there is a proper, Y-invariant subspace of V, which contains the vector v and on which γ is nondegenerate. Let us call such a representative *adapted* to the decomposition.

LEMMA 14. *Every cotype, which is not affine, is the sum of a unique indecomposable cotype, which is either the zero cotype or a nonzero* 1*-dimensional cotype, and a type.*

Proof. Let $(V, Y, v; \gamma)$ represent the nonaffine cotype ∇. Suppose that $v = 0$. Write $V = \{0\} \oplus V$. Then $\{0\}$ and V are Y-invariant, γ-orthogonal, and γ-nondegenerate. Hence ∇ is the sum of the zero cotype 0 and a type Δ, represented by $(Y, V; \gamma)$. Now suppose that $v \neq 0$. Because ∇ is not affine, v is not γ-isotropic, that is, $\gamma(v, v) = \varepsilon\,\alpha^2$, where $\varepsilon^2 = 1$ and $\alpha > 0$. Since $\mathrm{span}\{v\}$ is γ-nondegenerate, its orthogonal complement $\widetilde{V} = \mathrm{span}\{v\}^\gamma$ is also γ-nondegenerate. Let $\widetilde{\mathfrak{f}} = \{e_1, \ldots, e_n\}$ be a basis of $\widetilde{V}$ such that the matrix of $\widetilde{\gamma} = \gamma|\widetilde{V}$ is F. Then $\mathfrak{f} = \{e_1, \ldots, e_n, e_{n+1} = v\}$ is a basis of V such that the matrix of γ with respect to $\mathfrak{f}$ is $G = \begin{pmatrix} F & 0 \\ 0 & \varepsilon\alpha^2 \end{pmatrix}$. Since $Y \in \mathrm{o}(V, \gamma)$, the matrix of Y with respect to the basis $\mathfrak{f}$ is $Y = \begin{pmatrix} \widetilde{Y} & \varepsilon\alpha^2\,\widetilde{v} \\ -\widetilde{v}^T F & 0 \end{pmatrix}$, where $\widetilde{Y} \in \mathrm{o}(\widetilde{V}, \widetilde{\gamma})$ and $\widetilde{v} \in \widetilde{V}$. Thus the tuple $(V, Y, e_{n+1}; G)$ represents the cotype ∇. For every $w = \widetilde{w} + w_{n+1} e_{n+1} \in \widetilde{V} \oplus \mathrm{span}\{e_{n+1}\}$, the matrix of $L_{w, e_{n+1}}$ with respect to the basis $\mathfrak{f}$ is $\begin{pmatrix} 0 & \varepsilon\alpha^2\,\widetilde{w} \\ -\widetilde{w}^T F & 0 \end{pmatrix} \in \mathrm{o}(V, G)$, since

$$
\begin{aligned}
L_{w,e_{n+1}}(e_i) &= e*_{n+1}(e_i)w - w^*(e_i)e_{n+1} \\
&= -(w^T G e_i)e_{n+1} = -(\widetilde{w}^T F e_i)e_{n+1}, \qquad \text{for } 1 \le i \le n \\
L_{w,e_{n+1}}(e_{n+1}) &= e*_{n+1}(e_{n+1})w - w^*(e_{n+1})e_{n+1} \\
&= (e_{n+1}^T G e_{n+1})w - (w^T G e_{n+1})e_{n+1} \\
&= \varepsilon\alpha^2 (w - w_{n+1}e_{n+1}) = \varepsilon\alpha^2 \widetilde{w}.
\end{aligned}
$$

Therefore we may write $Y = \begin{pmatrix} \widetilde{Y} & 0 \\ 0 & 0 \end{pmatrix} + L_{\varepsilon\alpha^2 \widetilde{v}, e_{n+1}}$, which implies that the tuple $(V, Y, e_{n+1}; G)$ is equivalent to the tuple $(V, Y = \begin{pmatrix} \widetilde{Y} & 0 \\ 0 & 0 \end{pmatrix}, e_{n+1}; G)$. Now the subspace $\mathrm{span}\{e_{n+1}\}$ is G-nondegenerate, since the matrix of G restricted to $\mathrm{span}\{e_{n+1}\}$ is $(\varepsilon\alpha^2)$, which is nonzero. From $\check{Y}e_{n+1} = 0$, it follows that $\mathrm{span}\{e_{n+1}\}$ is $\check{Y}$-invariant. Clearly, the space $\widetilde{V} = \mathrm{span}\{e_{n+1}\}^G$ is also $\check{Y}$-invariant. Therefore the cotype ∇, represented by the tuple $(V, \check{Y}, e_{n+1}; G)$, is the sum of a 1-dimensional cotype $\widetilde{\nabla}$, represented by the tuple $(\mathrm{span}\{e_{n+1}\}, 0, e_{n+1}; (\varepsilon\alpha^2))$, and a type Δ, represented by $(\widetilde{Y}, \widetilde{V}; F)$. □

Lemma 15. *Every affine cotype can be written as a sum of an indecomposable affine cotype and a sum of indecomposable types. This decomposition is unique up to reordering of the summands which are types.*

Proof. Suppose that we are given an affine cotype ∇. Then ∇ is uniquely determined by its little cotype ∇_ℓ, where $\dim \nabla_\ell < \dim \nabla$. This correspondence respects decomposition: if ∇_ℓ is decomposable, then reconstructing ∇ as in the remark above, one finds that ∇ is decomposable. Conversely, if ∇ is decomposable, then using a representative adapted to a decomposition one finds that ∇_ℓ is decomposable. If ∇_ℓ is again affine, we look at its little cotype. Repeating this process a finite number of times, we obtain either the zero cotype and we stop or we obtain a nonzero cotype $\widehat{\nabla}$ which is not affine. By Lemma 14 $\widehat{\nabla}$ is a unique sum of a cotype $\widetilde{\nabla}$, which is either the zero cotype or a nonzero 1-dimensional cotype and a type Δ. By results of [3], the type Δ is a sum of indecomposable types, which is unique up to reordering the summands. This completes the proof of the lemma. □

PROPOSITION 16. *Let ∇ be an indecomposable affine cotype of dimension n. Then exactly one of the following alternatives holds.*

1. *n is even, say $n = 2h + 2$, $h \ge 0$. There is a representative $(V, Y, v; \gamma)$ of ∇ such that the following hold. There is a basis*

$$\{(-1)^h z, \ldots, -Y^{h-1}z, Y^h z; Y^h w, Y^{h-1}w, \ldots, w\} \tag{8}$$

of V, where $v = w$, $Y^{h+1} = 0$. With respect to the basis (8) the Gram matrix of γ is $\begin{pmatrix} 0 & I_{h+1} \\ I_{h+1} & 0 \end{pmatrix}$ and the matrix of Y is $\begin{pmatrix} -N^T & 0 \\ 0 & N \end{pmatrix}$, where $N = N_{h+1}$ is an $(h+1) \times (h+1)$ (upper) Jordan block $\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$. We use the notation $\nabla_n(0,0)$ for the cotype ∇.

2. *n is odd, say $n = 2m + 3$, $m \ge 0$. There is a representative $(V, Y, v; \gamma)$ of ∇ such that the following hold. There is a basis*

$$\{(-1)(\varepsilon/\mu^2)Y^{m+2}w, \ldots, (-1)^{m+1}(\varepsilon/\mu^2)Y^{2m+2}w; (1/\mu)Y^{m+1}w; Y^m w, \ldots, w\}, \tag{9}$$

where $\mu > 0$, $\varepsilon^2 = 1$, $v = w$, and $Y^n = 0$. We call μ a modulus of ∇. With respect to the basis (9) the Gram matrix of γ is $\begin{pmatrix} 0 & 0 & I_{m+1} \\ 0 & \varepsilon & 0 \\ I_{m+1} & 0 & 0 \end{pmatrix}$. The matrix of Y is $\begin{pmatrix} -N^T & -\epsilon\mu e_1 & 0 \\ 0 & 0 & \mu e_1^T \\ 0 & 0 & N \end{pmatrix}$, *where $N = N_{m+1}$ is an $(m+1) \times (m+1)$ upper Jordan block. Here $e_1^T = \overbrace{(1,0,\ldots,0)}^{m+1}$. Note that as a nilpotent matrix, Y has just one Jordan block. We use the notation $\nabla_n^{\varepsilon}(0),\ \mu$ for the cotype ∇.*

Proof. One easily checks that the given representatives do indeed define cotypes $\nabla_n(0,0)$ and $\nabla_n^{\varepsilon}(0),\ \mu$, respectively. Computing their little cotypes one finds that the little cotype of $\nabla_n^{\varepsilon}(0),\ \mu$ is $\nabla_{n-2}^{\varepsilon}(0),\ \mu$. And the little cotype of $\nabla_n(0,0)$ is $\nabla_{n-2}(0,0)$. Consider an indecomposable affine cotype of dimension n. As a cotype is uniquely determined by its little cotype, and this little cotype must thus also be indecomposable, it is either affine, and we may argue by induction, or it has dimension at most one and is described easily. □

Remark 17. It is noteworthy that we could choose the representatives in Proposition 16 to have nilpotent Y.

Remark 18. (The curious bijection) There is a curious bijection between the representatives that we choose here for indecomposable affine cotypes and the representatives that we used for indecomposable distinguished types in Proposition 5. The bijection preserves dimension, index, modulus, and Jordan type. It follows that we also get a bijection between affine cotypes and distinguished types with the same underlying $(V;\gamma)$. In other words, we get a bijection between adjoint orbits and coadjoint orbits for any affine orthogonal group.

7. Coadjoint Orbits of the Poincaré Group

In this section we use the theory of Section 6 to classify the coadjoint orbits of the Poincaré group $\mathrm{O}(4,2)_{e_5}$.

Let (V,γ) be a real vector space with a nondegenerate inner product γ of signature $(m,p) = (4,2)$. Suppose that the tuple $(V,Y',v;\gamma)$ represents an affine cotype in $\mathrm{O}(V,\gamma)$. Since $\mathrm{O}(V,\gamma)$ acts transitively on the collection of nonzero γ-isotropic vectors in V, there is a $P \in \mathrm{O}(V,\gamma)$ such that $Pv = e_5$. Hence the tuple $(V, Y = PY'P^{-1}, e_5;\gamma)$ is equivalent to $(V,Y,v;\gamma)$. Because e_5 is γ-isotropic and γ is nondegenerate on V, there is a γ-isotropic vector $e_0 \in V$ such that $\gamma(e_0,e_5) = 1$. In other words, $H = \mathrm{span}\{e_0,e_5\}$ is a hyperbolic plane in V. Because $\gamma|H$ is nondegenerate, we can extend $\{e_0,e_5\}$ to a γ-orthonormal basis $\mathfrak{e} = \{e_0,e_1,\ldots,e_4,e_5\}$ of V such that the matrix of γ with respect to $\mathfrak{e}$ is $K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & G & 0 \\ 1 & 0 & 0 \end{pmatrix}$, where $G^T = G$, $G^2 = I$, and G has signature $(3,1)$. Thus using the basis $\mathfrak{e}$ the tuple $(V,Y,e_5;\gamma)$ is the tuple $(V,Y,e_5;K)$.

Table 4 Possible $\mathfrak{o}(V,K)$-indecomposable affine cotypes

	Affine cotype	Dim	Index
1.	$\nabla_5^-(0),\ \mu$	5	3
2.	$\nabla_4(0,0)$	4	2
3.	$\nabla_3^-(0),\ \mu$	3	2
4.	$\nabla_3^+(0),\ \mu$	3	1
5.	$\nabla_2(0,0)$	2	1

Table 5 Possible $\mathrm{o}(V,K)$-indecomposable, which appear as a summand in the type Δ

	Type	Dim	Index
1.	$\Delta_1(\zeta, RP)$	4	2
2.	$\Delta_1^{\varepsilon}(\zeta, IP)$	4	2
3.	$\Delta_1(0,0)$	4	2
4.	$\Delta_2^+(0)$	3	2
5.	$\Delta_2^-(0)$	3	1
6.	$\Delta_0^-(\zeta, IP)$	2	2
7.	$\Delta_0(\zeta, RP)$	2	1
8.	$\Delta_0^+(\zeta, IP)$	2	0
9.	$\Delta_0^-(0)$	1	1
10.	$\Delta^+(0)$	1	0

Without loss of generality we can begin with an *affine* cotype ∇ in $\mathrm{o}(V,K)$ represented by the tuple

$$(\mathbb{R}^6, Y = \begin{pmatrix} y_0 & -\widetilde{x}^t G & 0 \\ \widetilde{y} & \widetilde{Y} & \widetilde{x} \\ 0 & -\widetilde{y}^T G & -y_0 \end{pmatrix}, e_5; K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & G & 0 \\ 1 & 0 & 0 \end{pmatrix}), \tag{10}$$

where $y_0 \in \mathbb{R}$, $\widetilde{x}, \widetilde{y} \in \mathbb{R}^4$, and $\widetilde{Y}^T G + G\widetilde{Y} = 0$, that is, $\widetilde{Y} \in \mathrm{o}(\mathbb{R}^4, G)$. By Proposition 15 we can write $\nabla = \widetilde{\nabla} + \Delta$, where the possible indecomposable affine cotypes $\widetilde{\nabla}$ in $\mathrm{o}(V,K)$ are listed in Table 4, and the possible indecomposable summands of the $\mathrm{o}(V,K)$ type Δ are listed in Table 5.

Therefore the possible decompositions of the affine cotype ∇ into a sum of an indecomposable affine cotype $\widetilde{\nabla}$ and a sum of indecomposable types is given in Table 6.

8. Normal Forms

We now give a table of explicit tuples $(\mathbb{R}^6, Y, e_5; K)$ which represent the corresponding affine cotypes listed in Table 6.

Table 6 Coadjoint orbits of the Poincaré group $\mathrm{O}(\mathbb{R}^6, K)_{e_5}$

	Indecomposable affine cotypes and sum of indecomposable types	Dim	Index
1.	$\nabla_5^-(0), \mu + \Delta_0^-(0)$	$5+1$	$3+1$
2.	$\nabla_4(0,0) + \Delta_0^-(\zeta, IP)$	$4+2$	$2+2$
3.	$\nabla_4(0,0) + \Delta_0^-(0) + \Delta_0^-(0)$	$4+2$	$2+2$
4.	$\nabla_3^-(0), \mu + \Delta_2^+(0)$	$3+3$	$2+2$
5.	$\nabla_3^-(0), \mu + \Delta_0^-(\zeta, IP) + \Delta_0^+(0)$	$3+3$	$2+2$
6.	$\nabla_3^-(0), \mu + \Delta_0(\zeta, RP) + \Delta_0^-(0)$	$3+3$	$2+2$
7.	$\nabla_3^-(0), \mu + \Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^+(0)$	$3+3$	$2+2$
8.	$\nabla_3^+(0), \mu + \Delta_0^-(\zeta, IP) + \Delta_0^-(0)$	$3+3$	$1+3$
9.	$\nabla_3^+(0), \mu + \Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^-(0)$	$3+3$	$1+3$
10.	$\nabla_2(0,0) + \Delta_2^+(0) + \Delta_0^-(0)$	$2+4$	$1+3$
11.	$\nabla_2(0,0) + \Delta_0^-(\zeta, IP) + \Delta_0(\zeta, RP)$	$2+4$	$1+3$
12.	$\nabla_2(0,0) + \Delta_0^-(\zeta, IP) + \Delta_0^-(0) + \Delta_0^+(0)$	$2+4$	$1+3$
13.	$\nabla_2(0,0) + \Delta_0(\zeta, RP) + \Delta_0^-(0) + \Delta_0^-(0)$	$2+4$	$1+3$
14.	$\nabla_2(0,0) + \Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^+(0)$	$2+4$	$1+3$

In our list of normal forms we use the following conventions. Let $\mathfrak{e} = \{e_0, e_1, \ldots, e_4, e_5\}$ be the *standard basis* for $\mathbb{R}^6$ such that the Gram matrix of the inner product is $K = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & G & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$, where $G = \left(\begin{smallmatrix} -I_3 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We call K the *standard form* of the inner product γ on $\mathbb{R}^6$ and G the *standard form* of the Lorentz inner product on $\mathbb{R}^4$ with standard basis $\widetilde{\mathfrak{e}} = \{e_1, \ldots, e_4\}$.

Thus with respect to the standard basis $\widetilde{\mathfrak{e}}$ the matrix of $\widetilde{Y} \in \mathrm{o}(\mathbb{R}^4, G)$ is

$$\begin{pmatrix} \widehat{z} & b \\ b^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & -z_3 & z_2 & b_1 \\ z_3 & 0 & -z_1 & b_2 \\ -z_2 & z_1 & 0 & b_3 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix},$$

where $b, z \in \mathbb{R}^3$.

In the list of normal forms below we give the matrix $\widetilde{Y}$ and vector v of the little cotype. We assume that given the little cotype, represented by $(\mathbb{R}^4, \widetilde{Y}, v; G)$, the normal form matrix $Y \in \mathrm{o}(\mathbb{R}^6, K)$ of the corresponding cotype, represented by $(\mathbb{R}^6, Y, e_5; K)$, is

$$Y = \left(\begin{array}{c|c|c} 0 & -v^T G & 0 \\ \hline 0 & \widetilde{Y} & v \\ \hline 0 & 0 & 0 \end{array}\right).$$

8.1. List of Representatives of the Affine Cotypes given in Table 6

1. **Affine cotype**: $\nabla_5^-(0), \mu + \Delta_0^-(0)$.

   Sum basis: $\{\mu^{-2}Y^3 w,\ -\mu^{-2}Y^4 w,\ \mu^{-1}Y^2 w,\ Yw,\ w;\ z\}$. *Conditions*: $Y^5 w = Yz = 0$; $\gamma(w, Y^4 w) = -\mu^2$, $\gamma(z,z) = -1$.

   Little cotype: *Normal form basis* $\widetilde{\mathfrak{e}}$:

   $$\{ \frac{1}{\sqrt{2}} (\mu^{-2}Y^3 w - Yw),\ \mu^{-1}Y^2 w,\ z,\ \frac{1}{\sqrt{2}}(\mu^{-2}Y^3 w + Yw)\}.$$

   Normal form matrix $\widetilde{Y}$ *and vector* v:

   $$\widetilde{Y} = \begin{pmatrix} -\frac{\mu}{\sqrt{2}}\widehat{e_3} & \frac{\mu}{\sqrt{2}} e_2 \\ \frac{\mu}{\sqrt{2}} e_2^T & 0 \end{pmatrix};\ v = \frac{1}{\sqrt{2}}(-e_1 + e_4).$$

2. **Affine cotype**: $\nabla_4(0,0) + \Delta_0^-(i\beta, \mathrm{IP})$.

   Sum basis: $\{-z,\ Yz,\ Yw, w;\ u,\ \beta^{-1}Yu\}$. *Conditions*: $Y^2 w = Y^2 z = 0$, $(Y^2 + \beta^2)\ u = 0$; $\gamma(Yz, w) = 1$ and $\gamma(u,u) = -1$.

   Little cotype: *Normal form basis* $\widetilde{\mathfrak{e}}$:

   $$\{\frac{1}{\sqrt{2}}(Yw + z),\ u,\ \beta^{-1}Yu,\ \frac{1}{\sqrt{2}}(Yw - z)\}.$$

Normal form matrix $\widetilde{Y}$ *and vector* v:

$$\widetilde{Y} = \begin{pmatrix} \beta\widehat{e_1} & 0 \\ 0 & 0 \end{pmatrix};\ \ v = \tfrac{1}{\sqrt{2}}(e_1 + e_4).$$

3. **Affine cotype**: $\nabla_4(0,0) + \Delta_0^-(0) + \Delta_0^-(0)$.

   Sum basis: $\{-z, Yz, Yw, w; u; v\}$. *Conditions*: $Y^2w = Y^2z = Yu = Yv = 0$; $\gamma(Yz, w) = 1$, $\gamma(u,u) = \gamma(v,v) = -1$.

   Little cotype: *Normal form basis* $\widetilde{\mathrm{e}}$:

$$\{\frac{1}{\sqrt{2}}(z + Yw), u,\ v,\ \frac{1}{\sqrt{2}}(Yw - z)\}.$$

   Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y} = 0$; $v = \frac{1}{\sqrt{2}}(e_1 + e_4)$.

4. **Affine cotype**: $\nabla_3^-(0), \mu + \Delta_2^+(0)$.

   Sum basis: $\{\mu^{-2}Y^2w,\ \mu^{-1}Yw,\ w;\ u,\ Yu,\ Y^2u\}$.*Conditions*: $Y^3w = Y^3u = 0$; $\gamma(w, Y^2w) = \mu^2$, $\gamma(u, Y^2u) = 1$.

   Little cotype: *Normal form basis* $\widetilde{\mathrm{e}}$:

$$\{\mu^{-1}Yw,\ \frac{1}{\sqrt{2}}(u - Y^2u),\ Yu,\ \frac{1}{\sqrt{2}}(u + Y^2u)\}.$$

   Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y} = \begin{pmatrix} \frac{1}{\sqrt{2}}\widehat{e_1} & \frac{1}{\sqrt{2}}e_3 \\ \frac{1}{\sqrt{2}}e_3^T & 0 \end{pmatrix}$; $v = \mu\, e_1$.

5. **Affine cotype**: $\nabla_3^-(0), \mu + \Delta_0^-(i\beta, IP) + \Delta_0^+(0)$.

   Sum basis: $\{\mu^{-2}Y^2w,\ \mu^{-1}Yw,\ w;\ u,\ \beta^{-1}Yu;\ v\}$. *Conditions*: $Y^3w = Yv = 0$, and $(Y^2 + \beta^2)u = 0$; $\gamma(w, Y^2w) = \mu^2$, $\gamma(u,u) = -1$, and $\gamma(v,v) = 1$.

   Little cotype: *Normal form basis* $\widetilde{\mathrm{e}}$: $\{\mu^{-1}Yw,\ u,\ \beta^{-1}Yu,\ v\}$.
   Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y} = \begin{pmatrix} \beta\widehat{e_1} & 0 \\ 0 & 0 \end{pmatrix}$; $v = \mu\, e_1$.

6. **Affine cotype**: $\nabla_3^-(0), \mu + \Delta_0(\alpha, RP) + \Delta_0^-(0)$.

   Sum basis: $\{\mu^{-2}Y^2w,\ \mu^{-1}Yw,\ w;\ u,\ \alpha^{-1}Yu;\ v\}$. *Conditions*: $Y^3w = Yv = 0$, and $(Y^2 - \alpha^2)u = 0$; $\gamma(w, Y^2w) = \mu^2$, $\gamma(u,u) = 1$, and $\gamma(v,v) = -1$.

   Little cotype: *Normal form basis* $\widetilde{\mathrm{e}}$: $\{\mu^{-1}Yw,\ \alpha^{-1}Yu,\ v,\ u;\ w\}$.
   Normal form matrix $\widetilde{Y}$ *and vector* v : $\widetilde{Y} = \begin{pmatrix} 0 & \alpha e_2 \\ \alpha e_2^T & 0 \end{pmatrix}$; $v = \mu\, e_1$.

7. **Affine cotype**: $\nabla_3^-(0), \mu + \Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^+(0)$.

   Sum basis: $\{\mu^{-2}Y^2w,\ \mu^{-1}Yw,\ w;\ u;\ v;\ z\}$. *Conditions*: $Y^3w = Yu = Yv = Yz = 0$; $\gamma(w, Y^2w) = \mu^2$, $\gamma(u,u) = \gamma(v,v) = -1$, $\gamma(z,z) = 1$.

Little cotype: *Normal form basis* $\widetilde{e}$:$\{\mu^{-1}Yw,\ u,\ v,\ z\}$.
Normal form matrix $\widetilde{Y}$ and vector v: $\widetilde{Y}=0$; $v=\mu e_1$.

8. **Affine cotype**: $\nabla_3^+(0), \mu+\Delta_0^-(i\beta, IP)+\Delta_0^-(0)$.

Sum basis: $\{-\mu^{-2}Y^2w,\ \mu^{-1}Yw,\ w;\ u,\ \beta^{-1}Yu;\ v\}$ *Conditions*: $Y^3w=Yv=0$ and $(Y^2+\beta^2)u=0$; $\gamma(w,Y^2w)=-\mu^2$, and $\gamma(u,u)=\gamma(v,v)=-1$.

Little cotype: *Normal form basis* $\widetilde{e}$: $\{u, \beta^{-1}Yu,\ v,\ \mu^{-1}Yw\}$.
Normal form matrix $\widetilde{Y}$ *and vector* v : $\widetilde{Y}=\begin{pmatrix}\beta\widehat{e}_3 & 0\\ 0 & 0\end{pmatrix}$; $v=\mu e_4$.

9. **Affine cotype**: $\nabla_3^+(0), \mu+\Delta_0^-(0)+\Delta_0^-(0)+\Delta_0^-(0)$.

Sum basis: $\{-\mu^{-2}Y^2w,\ \mu^{-1}Yw,\ w;\ u;\ v;\ z\}$. *Conditions*: $Y^3w=Yu=Yv=Yz=0$; $\gamma(w,Y^2w)=-\mu^2$, and $\gamma(u,u)=\gamma(v,v)=\gamma(z,z)=-1$.

Little cotype: *Normal form basis* $\widetilde{e}$: $\{u,\ v,\ z,\ \mu^{-1}Yw\}$.
Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y}=0$; $v=\mu e_4$.

10. **Affine cotype**: $\nabla_2(0,0)+\Delta_2^+(0)+\Delta_0^-(0)$.

Sum basis: $\{z,\ w;\ Y^2u,\ Yu,\ u;\ v\}$. *Conditions*: $Y^3u=Yw=Yv=Yz=0$; $\gamma(z,w)=1$, $\gamma(u,Y^2u)=1$, and $\gamma(v,v)=-1$.

Little cotype: *Normal form basis* $\widetilde{e}$:

$$\{\frac{1}{\sqrt{2}}(u-Y^2u),\ Yu,\ v,\ \frac{1}{\sqrt{2}}(u+Y^2u)\}.$$

Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y}=\begin{pmatrix}\frac{1}{\sqrt{2}}\widehat{e}_3 & \frac{1}{\sqrt{2}}e_2\\ \frac{1}{\sqrt{2}}e_2^T & 0\end{pmatrix}$; $v=0$.

11. **Affine cotype**: $\nabla_2(0,0)+\Delta_0^-(i\beta, IP)+\Delta_0(\alpha, RP)$.

Sum basis: $\{z,\ w;\ u,\ \beta^{-1}Yu;\ v,\ \alpha^{-1}Yv\}$. *Conditions*: $Yz=Yw=0$, $(Y^2+\beta^2)u=0$, $(Y^2-\alpha^2)v=0$; $\gamma(z,w)=\gamma(v,v)=1$, and $\gamma(u,u)=-1$.

Little cotype: *Normal form basis* $\widetilde{e}$:

$$\{u,\ \beta^{-1}Yu,\ \alpha^{-1}Yv,\ v\}.$$

Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y}=\begin{pmatrix}\beta\widehat{e}_3 & \alpha e_3\\ \alpha e_3^T & 0\end{pmatrix}$; $v=0$.

12. **Affine cotype**: $\nabla_2(0,0)+\Delta_0^-(i\beta, IP)+\Delta_0^-(0)+\Delta_0^+(0)$.

Sum basis: $\{z,\ w;\ u,\ \beta^{-1}Yu;\ v;\ y\}$. *Conditions*: $Yz=Yw=Yv=Yy=0$; $\gamma(z,w)=\gamma(y,y)=1$ and $\gamma(u,u)=\gamma(v,v)=-1$.

Little cotype: *Normal form basis* $\widetilde{e}$: $\{u,\ \beta^{-1}Yu,\ v;\ y;\ w\}$.

Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y} = \begin{pmatrix} \beta\widehat{e}_3 & 0 \\ 0 & 0 \end{pmatrix}$; $v = 0$.

13. **Affine cotype**: $\nabla_2(0,0) + \Delta_0(\alpha, RP) + \Delta_0^-(0) + \Delta_0^-(0)$.

    Sum basis: $\{z, w; u, \alpha^{-1}Yu; v; y\}$. *Conditions*: $Yz = Yw = Yv = Yy = 0$, $(Y^2 - \alpha^2)u = 0$; $\gamma(z,w) = \gamma(u,u) = 1$ and $\gamma(v,v) = \gamma(y,y) = -1$.

    Little cotype: *Normal form basis* $\widetilde{\mathrm{e}}$: $\{\alpha^{-1}Yu, v, y, u\}$.
    Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y} = \begin{pmatrix} 0 & \alpha e_1 \\ \alpha e_1^T & 0 \end{pmatrix}$; $v = 0$.

14. **Affine cotype**: $\nabla_2(0,0) + \Delta_0^-(0) + \Delta_0^-(0) + \Delta_0^+(0)$.

    Sum basis: $\{z, w; u; v; y; s\}$. *Conditions*: $Yz = Yw = Yv = Yy = Ys = 0$; $\gamma(z,w) = \gamma(s,s) = 1$ and $\gamma(u,u) = \gamma(v,v) = \gamma(y,y) = -1$.

    Little cotype: *Normal form basis* $\widetilde{\mathrm{e}}$: $\{v, u, y, s\}$.
    Normal form matrix $\widetilde{Y}$ *and vector* v: $\widetilde{Y} = 0$; $v = 0$.

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