On a Grauert-Riemenschneider vanishing theorem for Frobenius split varieties in characteristic p.

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1 Introduction

It is known that the Grauert-Riemenschneider vanishing theorem is not valid in characteristic p ([1]). Here we show that it may be restored in the presence of a suitable Frobenius splitting. The proof uses interchanging two projective limits, one involving iterated Frobenius maps, cf. [2] and [4], the other coming from Grothendieck's theorem on formal functions. That leads to the following general vanishing theorem which we then apply in the situation of the Grauert-Riemenschneider theorem.

Theorem 1.1 *Let $\pi : X \to Y$ be a proper morphism of schemes of finite type over a perfect field of characteristic $p > 0$. Let D be a closed subscheme of X with ideal sheaf $\mathcal{I}$, let E be a closed subscheme of Y and let $i \geq 0$ such that*
1. D contains the geometric points of $\pi^{-1}E$.
2. $R^i\pi_*(\mathcal{I})$ vanishes off E.
3. X is Frobenius split, compatibly with D.
Then $R^i\pi_*(\mathcal{I})$ vanishes on all of Y.

Theorem 1.2 (Grauert-Riemenschneider with Frobenius splitting.) *Let $\pi : X \to Y$ be a proper birational morphism of varieties in characteristic $p > 0$ such that:*
1. X is non-singular and there is $\sigma \in H^0(X, K_X^{-1}) = H^0(X, c_1(X))$ such that

σ^{p-1} splits X. (cf. [5].)
2. $D = \mathrm{div}(\sigma)$ contains the exceptional locus of π set theoretically.
Then $R^i\pi_*K_X = 0$ for $i > 0$.

Remark 1.3 It will be clear from the proof that many variations on our Grauert-Riemenschneider theorem are possible. For instance, one may replace D by some subdivisor which still contains the exceptional locus, and thus replace K_X in the conclusion by the new $\mathcal{O}_X(-D)$. Similarly, the birationality assumption may be weakened, as it is used only to conclude that condition 2 of 1.1 is satisfied.

2 Proofs

2.1 Proof of 1.2. We assume theorem 1.1. For E we take the image of the exceptional locus. Dualizing σ we get a short exact sequence

$$0 \to K_X \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

so K_X may be identified with the ideal sheaf $\mathcal{I}$ of D. That D is compatibly split is clear from local computations, cf. Remark on page 36 of [5]. □

Lemma 2.2 *Let $\cdots \to M_2 \to M_1 \to M_0$ be a projective system of artinian modules over some ring R, with transition maps $f_i^j : M_j \to M_i$. If f_0^i is nonzero for all i, then the projective limit is nonzero.*

Proof. Put $M_i^{\mathrm{stab}} = \bigcap_{j \geq i} f_i^j(M_j)$. Then $M_i^{\mathrm{stab}} = f_i^k(M_k)$ for $k \gg 0$. So

$$f_i^{i+1}(M_{i+1}^{\mathrm{stab}}) = f_i^{i+1} f_{i+1}^k(M_k) = f_i^k(M_k) = M_i^{\mathrm{stab}}$$

for $k \gg 0$. Therefore we have a subsystem (M_i^{stab}) with nonzero surjective maps, whence the result.

2.3 Proof of 1.1. We argue by contradiction. We may assume Y is affine, so that $R^i\pi_*(\mathcal{I})$ equals $H^i(X, \mathcal{I})$. Choose an irreducible component, with generic point y say, of the support on Y of $H^i(X, \mathcal{I})$, which we suppose to be nonzero. Observe that $y \in E$. The Frobenius map F as well as its splitting act on the exact sequence of sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Therefore the Frobenius and its iterates act by split injective endomorphisms, p-linear over $A = \Gamma(Y, \mathcal{O}_Y)$, on $H^i(X, \mathcal{I})$, and the same remains true after localisation and completion at y. Let R be a regular ring of the form $L[[X_1, \ldots, X_m]]$ mapping onto $A^\wedge_y$, where L is a field of representatives in the completed local ring $A^\wedge_y$. In the projective system of artinian modules

$$\cdots R \otimes^{p^r} H^i(X, \mathcal{I})^\wedge_y \to R \otimes^{p^{r-1}} H^i(X, \mathcal{I})^\wedge_y \to \cdots$$

all maps towards $R \otimes^{p^0} H^i(X, \mathcal{I})^\wedge_y = H^i(X, \mathcal{I})^\wedge_y$ are nonzero. Here $R \otimes^{p^r}$ refers to base change along the r times iterated Frobenius endomorphism of the regular ring R, and the projective system is thus the one defining the "leveling" $G(H^i(X, \mathcal{I})^\wedge_y)$, in the sense of [4], of $H^i(X, \mathcal{I})^\wedge_y$ as an R module. The projective limit is nonzero by the Lemma. On the other hand, as R is a finite free module over R via F^r, one may also compute

$$G(H^i(X, \mathcal{I})^\wedge_y) = \lim_{\leftarrow r} R \otimes^{p^r} H^i(X, \mathcal{I})^\wedge_y$$

as follows

$$\lim_{\leftarrow r} R \otimes^{p^r} H^i(X, \mathcal{I})^\wedge_y = \lim_{\leftarrow r} R \otimes^{p^r} \lim_{\leftarrow s} H^i(X_s, \mathcal{I}_s) =$$

$$\lim_{\leftarrow r} \lim_{\leftarrow s} R \otimes^{p^r} H^i(X_s, \mathcal{I}_s) = \lim_{\leftarrow s} \lim_{\leftarrow r} R \otimes^{p^r} H^i(X_s, \mathcal{I}_s)$$

where X_s and $\mathcal{I}_s$ are the usual thickenings from Grothendieck's theorem on formal functions. But by the Artin-Rees lemma the Frobenius map acts nilpotently on $\mathcal{I}_s$, (note that some power of $\mathcal{I}$ is contained in the pull back of the ideal sheaf of E), so $\lim_{\leftarrow r} R \otimes^{p^r} H^i(X_s, \mathcal{I}_s)$ vanishes. But then $G(H^i(X, \mathcal{I})^\wedge_y)$ is both nonzero and zero. □

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