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The K_2's of a 2-dimensional regular local ring and its quotient field

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Theorem. Let R be a (noetherian) 2-dimensional regular local ring with quotient field F. Then the natural map $K_2(R) \to K_2(F)$ is an injection.

Proof. Let x,y be a regular system of parameters in R.

Lemma 1. $K_2(R) \to K_2(R_x)$ is injective.

Proof. Quillen's localization sequence ([1], §7, 3.2) $\dots \to K_2(R/xR) \overset{\alpha}{\to} K_2(R) \to K_2(R_x) \to \dots$ is exact, so we have to show that α is zero.The composition $K_2(R) \overset{\beta}{\to} K_2(R/xR) \overset{\alpha}{\to} K_2(R)$ is zero, because it amounts to multiplication by $[R/xR] = 0 \in K_0(R)$. ([1], §3). So it is sufficient to show that β is surjective. Now $K_2(R/xR)$ is generated by symbols ([2], Thm. 2.7), and units lift from R/xR to R, so β is indeed surjective. □

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The ring R/xR is a discrete valuation ring, so we can put:

$V_i = \{g \in R |$ valuation of g/xR in R/xR is at most $i\}$.

S_i = multiplicative system generated by x and V_i .

For example: $x \in S_0$, $y \in V_1$.

Lemma 2. Let S be a multiplicative system in R , containing S_i , and let $f \in V_{i+1}$ be such that $1/f \notin S^{-1}R$. Then $K_2(S^{-1}R) \to K_2(S^{-1}R_f)$ is injective.

Proof. We proceed as in the proof of Lemma 1. As R is a UFD one easily sees that R/fR is a local domain. It has dimension 1, so $R_x/fR_x = (R/fR)_{x/fR}$ is a field k , equal to $S^{-1}R/fS^{-1}R$. Again $K_2(k)$ is generated by symbols and we will be done if we can lift units from k to $S^{-1}R$. We can lift x/fR to the unit x , so we only need to consider $\bar{u} = u/fR \in R/fR$ with $\bar{u} \neq 0$, $\bar{u}$ not divisible by x/fR in R/fR . Write $u/xR = af/xR + b/xR$ with $b \in V_i$. Then $u = af + b + cx$ with $b + cx \in V_i \subseteq S$. So u/fR can be lifted to the unit $b + cx$ in $S^{-1}R$. □

Corollary. $K_2(S_i^{-1}R) \to K_2(S_{i+1}^{-1}R)$ is injective for $i \geq 0$.

Proof. Pass to the direct limit or note that only finitely many elements of $S_{i+1}^{-1}R$ are needed to prove that something is in the kernel. □

The Theorem follows from the Corollary.

Remark. This proof generalizes a known proof for dimension one.

References.

[1] D. Quillen, Higher algebraic K-theory I, Lecture Notes in Mathematics 341, 85-147, Springer 1973.

[2] M.R. Stein and R.K. Dennis, K_2 of radical ideals and semilocal rings revisited, Lecture Notes in Mathematics 342, 281-303, Springer 1973.

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