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Cohomological finite generation for the group scheme SL_2

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ABSTRACT

Let G be the group scheme SL_2 defined over a noetherian ring $\mathbf{k}$. If G acts on a finitely generated commutative $\mathbf{k}$-algebra A, then $H^*(G, A)$ is a finitely generated $\mathbf{k}$-algebra.

1. Introduction

Let $\mathbf{k}$ be a noetherian ring. Consider a flat linear algebraic group scheme G defined over $\mathbf{k}$. Recall that G has the cohomological finite generation property (CFG) if the following holds: Let A be a finitely generated commutative $\mathbf{k}$-algebra on which G acts rationally by $\mathbf{k}$-algebra automorphisms. (So G acts from the right on Spec(A).) Then the cohomology ring $H^*(G, A)$ is finitely generated as a $\mathbf{k}$-algebra. Here, as in [3, I.4], we use the cohomology introduced by Hochschild, also known as 'rational cohomology'.

This note is part of the project of studying (CFG) for reductive G. Recall that the breakthrough of Touzé [4] settled the case when $\mathbf{k}$ is a field [7]. And [8, Theorem 10.1] extended this to the case that $\mathbf{k}$ contains a field. In this paper we show that in the case $G = SL_2$ one can dispense with the condition that $\mathbf{k}$ contains a field. According to the last item of [8, Theorem 10.5] it suffices to show that $H^*(G, A/pA)$ is a noetherian module over $H^*(G, A)$ whenever p is a prime number. We fix p. To prove the noetherian property we employ universal cohomology classes as in earlier work. More specifically, we lift the cohomology classes $c_r[a]^{(j)}$ of [5, 4.6] to classes in cohomology of SL_2 over the integers with flat coefficient module $\Gamma^m \Gamma^{p^{r+j}}(\mathfrak{gl}_2)$. We get the lifts with explicit formulas that do not seem to generalize to SL_n with $n > 2$. Once we have the lifts of the cohomology classes we can lift enough of the mod p constructions to conclude that $H^*(G, A)$ hits much of $H^*(G, A/pA)$.

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As $H^*(G, A/pA)$ itself is a finitely generated **k**-algebra this will then imply that $H^*(G, A/pA)$ is a noetherian module over $H^*(G, A)$.

For simplicity of reference we use [8]. As we are working with SL_2 that amounts to serious overkill. For instance, the work of Touzé is not needed for SL_2. Further the 'functorial resolution of the ideal of the diagonal in a product of Grassmannians' now just means that the ideal sheaf of the diagonal divisor in a product of two projective lines is the familiar line bundle $\mathcal{O}(-1) \boxtimes \mathcal{O}(-1)$. And Kempf vanishing for SL_2 is immediate from the computation of the cohomology of line bundles on $\mathbb{P}^1$.

2. Rank one

We take $G = SL_2$ as group scheme over the noetherian ring **k**. Initially **k** is just $\mathbb{Z}$. Let T be the diagonal torus and B the Borel subgroup of lower triangular matrices. Its root α is the negative root.

2.1. *Cocycles for the additive group*

We have fixed a prime p. Define $\Phi(X, Y) \in \mathbb{Z}[X, Y]$ by

$$(X + Y)^p = X^p + Y^p + p\Phi(X, Y).$$

By induction one gets for $r \geqslant 1$

$$(X + Y)^{p^r} \equiv X^{p^r} + Y^{p^r} + p\Phi\bigl(X^{p^{r-1}}, Y^{p^{r-1}}\bigr) \mod p^2.$$

Put

$$c_r^{\mathbb{Z}}(X, Y) = \frac{(X + Y)^{p^r} - X^{p^r} - Y^{p^r}}{p} \in \mathbb{Z}[X, Y].$$

We think of $c_r^{\mathbb{Z}}$ as a 2-cochain in the Hochschild complex $C^\bullet(\mathbb{G}_a, \mathbb{Z})$ as treated in [3, I 4.14, I 4.20]. Then $c_r^{\mathbb{Z}}$ is a 2-cocycle because $pc_r^{\mathbb{Z}}$ is a coboundary. One has

$$c_r^{\mathbb{Z}}(X, Y) \equiv \Phi\bigl(X^{p^{r-1}}, Y^{p^{r-1}}\bigr) \mod p.$$

Taking cup products one finds a $2m$-cocycle $c_r^{\mathbb{Z}}(X, Y)^{\cup m}$ representing a class in $H^{2m}(\mathbb{G}_a, \mathbb{Z})$. The cocycle $c_r^{\mathbb{Z}}$ lifts the $(r-1)$-st Frobenius twist of the Witt vector class that was our starting point in [5, Section 4]. Now our strategy will be to follow [5, Section 4], lifting all relevant mod p constructions to the integers. That will do the trick.

2.2. *Universal classes*

Our next task is to construct a universal class $c_r[m]^{(j)}$ in $H^{2mp^{r-1}}(G, \Gamma^m \Gamma^{p^{r+j}}(\mathfrak{gl}_2))$.

Let $r \geqslant 1$, $j \geqslant 0$, $m \geqslant 1$. Let α be the negative root, and let $x_\alpha : \mathbb{G}_a \to SL_2$ be its root homomorphism, with image U_α. For a $\mathbb{Z}$-module V its m-th module of divided powers is written as $\Gamma^m V$ and its dual $\operatorname{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$ is written as $V^\#$.

Consider the representation $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ of G with its restriction $x_\alpha^* \Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ to $\mathbb{G}_a$. Its lowest weight is $mp^{r+j}\alpha$. Say e_α is the elementary matrix $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ that spans the α weight space of $\mathfrak{gl}_2$, and $e_\alpha^{[mp^{r+j}]}$ denotes its divided power in $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$. Then $c_{j+1}^{\mathbb{Z}}(X, Y)^{\cup mp^{r-1}} e_\alpha^{[mp^{r+j}]}$ represents a class in $H^{2mp^{r-1}}(\mathbb{G}_a, x_\alpha^* \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ and the corresponding element of $H^{2mp^{r-1}}(U_\alpha, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ is T-invariant. So we get a class in $H^{2mp^{r-1}}(B, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ and by Kempf vanishing ([3, II B.3] with $\lambda = 0$) a class in $H^{2mp^{r-1}}(G, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$. Recall that one obtains a natural map from

$\Gamma^{p^{r+j}}(\mathfrak{gl}_2 \bmod p)$ to the $(r+j)$-th Frobenius twist $(\mathfrak{gl}_2 \bmod p)^{(r+j)}$ by dualizing the map from $(\mathfrak{gl}_2^\# \bmod p)$ to $S^{p^{r+j}}(\mathfrak{gl}_2^\# \bmod p)$ that raises a vector $v \in (\mathfrak{gl}_2^\# \bmod p)$ to its p^{r+j}-th power. So $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ maps naturally to $\Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r+j)})$ by way of $\Gamma^m\Gamma^{p^{r+j}}(\mathfrak{gl}_2)$. Applying this to our class in $H^{2mp^{r-1}}(G, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ we hit a class in $H^{2mp^{r-1}}(G, \Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r+j)}))$, which is where $c_r[m]^{(j)}$ of [5, 4.6] lives. On the root subgroup U_α mod p it is given by the cocycle $\Phi(X^{p^j}, Y^{p^j})^{\cup mp^{r-1}} e_\alpha^{(r+j)[m]}$ mod p, where $e_\alpha^{(r+j)[m]}$ mod p is our notation for the obvious basis vector of the lowest weight space of $\Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r+j)})$. This cocycle is the same as the one used in [5, 4.6] to construct $c_r[m]^{(j)}$. But then their cohomology classes agree on B and G also. So we have lifted the $c_r[m]^{(j)}$ of [5, 4.6] to a cohomology group with a coefficient module $\Gamma^m\Gamma^{p^{r+j}}(\mathfrak{gl}_2)$ that is flat over the integers.

Notation 2.3. Simply write $c_r[m]^{(j)}$ for the lift in $H^{2mp^{r-1}}(G, \Gamma^m\Gamma^{p^{r+j}}(\mathfrak{gl}_2))$.

2.4. Pairings

In [5, 4.7] we used the pairing between the modules $\Gamma^m(\mathfrak{gl}_2 \bmod p)^{(r)}$ and $S^m(\mathfrak{gl}_2^\# \bmod p)^{(r)}$. We want to lift it to a pairing between representations $\Gamma^m(X_r)$ and $S^m(Y_r)$ of G over $\mathbb{Z}$. We take $X = X_r = \Gamma^{p^r}(\mathfrak{gl}_2)$ and define $K = \ker(X \to (\mathfrak{gl}_2 \bmod p)^{(r)})$.

Put $Y = Y_r = \ker(\mathrm{Hom}_\mathbb{Z}(X, \mathbb{Z}) \to \mathrm{Hom}_\mathbb{Z}(K, \mathbb{Z}/p\mathbb{Z}))$. Then $Y \to \mathrm{Hom}_\mathbb{Z}((X/K), \mathbb{Z}/p\mathbb{Z})$ is surjective because X is a free $\mathbb{Z}$-module. Notice that $\mathrm{Hom}_\mathbb{Z}((X/K), \mathbb{Z}/p\mathbb{Z})$ is just $(\mathfrak{gl}_2^\# \bmod p)^{(r)}$. Thus Y_r is flat and maps onto $(\mathfrak{gl}_2^\# \bmod p)^{(r)}$.

We have a commutative diagram

$$\begin{array}{ccc} \Gamma^m X \otimes S^m Y & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r)}) \otimes S^m((\mathfrak{gl}_2^\# \bmod p)^{(r)}) & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \end{array}$$

and the left vertical arrow is surjective. So we have found our lift of the pairing from [5, 4.7].

Remark 2.5. Notice that we do not use the precise shape of X here. What matters is that X is free over $\mathbb{Z}$, with a surjection of G modules $X \to (\mathfrak{gl}_2 \bmod p)^{(r)}$, and that, for $1 \leqslant i \leqslant r$, we have an element in $H^{2mp^{i-1}}(G, \Gamma^m X)$, suggestively denoted by $c_i[m]^{(r-i)}$, that is mapped to the $c_i[m]^{(r-i)}$ of [5] under the map induced by $X \to (\mathfrak{gl}_2 \bmod p)^{(r)}$.

2.6. Noetherian base ring

From now on let $\mathbf{k}$ be an arbitrary commutative noetherian ring. By base change to $\mathbf{k}$ we get a group scheme over $\mathbf{k}$ that we write again as $G = SL_2$. We simply write X_r for $X_r \otimes_\mathbb{Z} \mathbf{k}$ and we write Y_r for $Y_r \otimes_\mathbb{Z} \mathbf{k}$. We keep suppressing the base ring $\mathbf{k}$ in most notations, so that $X_r = \Gamma^{p^r}(\mathfrak{gl}_2)$, with classes $c_i[m]^{(r-i)}$ in $H^{2mp^{i-1}}(G, \Gamma^m X_r)$. The commutative diagram above becomes after base change

$$\begin{array}{ccc} \Gamma^m X_r \otimes S^m Y_r & \longrightarrow & \mathbf{k} \\ \downarrow & & \downarrow \\ \Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r)}) \otimes S^m((\mathfrak{gl}_2^\# \bmod p)^{(r)}) & \longrightarrow & \mathbf{k} \bmod p \end{array}$$

Lemma 2.7. *If V is a representation of G and $v \in V$, then the subrepresentation generated by v exists and is finitely generated as a k-module.*

Proof. As $\mathbf{k}[G]$ is a free $\mathbf{k}$-module, this follows from [6, Exposé VI, Lemme 11.8]. □

2.8. Cup products from pairings

Let U, V, W, Z be G-modules, and $\phi : U \otimes V \to Z$ a G-module map. We call ϕ a pairing. Computing with Hochschild complexes one gets cup products $H^i(G,U) \otimes H^j(G, V \otimes W) \to H^{i+j}(G, Z \otimes W)$ induced by ϕ. Note that we are not assuming that the modules are flat over $\mathbf{k}$. We think of the Hochschild complex for computing $H^i(G,M)$ as $(C^*(G,\mathbf{k}[G]) \otimes M)^G$, where $C^*(G,\mathbf{k}[G])$ has a differential graded algebra structure as described in [7, Section 6.3].

2.9. Hitting invariant classes

Definition 2.10. Recall that we call a homomorphism of $\mathbf{k}$-algebras $f : A \to B$ *noetherian* if f makes B into a noetherian left A-module. It is called *power surjective* [2, Definition 2.1] if for every $b \in B$ there is an $n \geqslant 1$ so that the power b^n is in the image of f.

See [7, Section 6.2] for some relevant properties of noetherian maps in cohomology. We are now going to look for noetherian maps. We keep the prime p fixed. Let $r \geqslant 1$. Let $\bar{G}$ denote G base changed to ($\mathbf{k}$ mod p), and let $\bar{G}_r$ denote its r-th Frobenius kernel. More specifically, take the Frobenius kernel $(G_r)_{\mathbb{Z}/p\mathbb{Z}}$ of $(SL_2)_{\mathbb{Z}/p\mathbb{Z}}$ and let $\bar{G}_r$ be obtained from $(G_r)_{\mathbb{Z}/p\mathbb{Z}}$ by base change $\mathbb{Z}/p\mathbb{Z} \to \mathbf{k}$ mod p. Now $(SL_2)_{\mathbb{Z}/p\mathbb{Z}}/(G_r)_{\mathbb{Z}/p\mathbb{Z}}$ is affine, and quotients remain affine under base change, cf. [3, I.5.5(1), I.5.4(5)], so $\bar{G}/\bar{G}_r$ is affine. Thus $\bar{G}_r$ is a ($\mathbf{k}$ mod p)-flat exact normal subgroup scheme of $\bar{G}$ [3, I 6.5], and we have a Hochschild–Serre spectral sequence as in [3, I 6.6] for $\bar{G}_r$ in $\bar{G}$. We use bars to indicate structures having ($\mathbf{k}$ mod p) as base ring. Let $\bar{C}$ be a finitely generated commutative ($\mathbf{k}$ mod p)-algebra with $\bar{G}$ action on which $\bar{G}_r$ acts trivially. By [2, Remark 52] we may view $\bar{C}$ also as an algebra with G action. Let $\mathcal{C}$ be a finitely generated commutative $\mathbf{k}$-algebra with G action and let $\pi : \mathcal{C} \to \bar{C}$ be a power surjective equivariant homomorphism.

Theorem 2.11. *$H^{\mathrm{even}}(G,\mathcal{C}) \to H^0(G, H^*(\bar{G}_r,\bar{C}))$ is noetherian.*

Proof. By [1, Theorem 1.5, Remark 1.5.1] $H^*(\bar{G}_r,\bar{C})$ is a noetherian module over the finitely generated graded algebra

$$\bar{R} = \bigotimes_{a=1}^{r} S^*\big((\bar{\mathfrak{gl}}_2^{(r)})^{\#}(2p^{a-1})\big) \otimes \bar{C}.$$

Here $(\bar{\mathfrak{gl}}_2^{(r)})^{\#}(2p^{a-1})$ means that one places a copy of $(\bar{\mathfrak{gl}}_2^{(r)})^{\#}$ in degree $2p^{a-1}$. It is easy to see that the obvious map from $\mathcal{R} = \bigotimes_{a=1}^{r} S^*(Y_r(2p^{a-1})) \otimes \mathcal{C}$ to $\bar{R}$ is noetherian. So by invariant theory [2, Theorem 9], $H^0(G, H^*(\bar{G}_r,\bar{C}))$ is a noetherian module over the finitely generated algebra $H^0(G,\mathcal{R})$. By [7, Remark 6.7] it now suffices to factor the map $H^0(G,\mathcal{R}) \to H^0(G, H^*(\bar{G}_r,\bar{C}))$ as a set map through $H^{\mathrm{even}}(G,\mathcal{C}) \to H^0(G, H^*(\bar{G}_r,\bar{C}))$.

On a summand

$$H^0\left(G, \bigotimes_{a=1}^{r} S^{i_a}\big(Y_r(2p^{a-1})\big) \otimes \mathcal{C}\right)$$

of $H^0(G,\mathcal{R})$ we simply take cup product with the (lifted) $c_a[i_a]^{(r-a)}$ according to the pairing of $S^{i_a}(Y_r)$ with $\Gamma^{i_a}(X_r) = \Gamma^{i_a}\Gamma^{p^r}(\mathfrak{gl}_2)$. In the proof of [5, Corollary 4.8] one has a similar description of the map to $H^*(\bar{G}_r,\bar{C})$ on the summand

$$H^0\left(G, \bigotimes_{a=1}^{r} S^{i_a}\left(\left(\bar{\mathfrak{gl}}_2^{(r)}\right)^{\#}(2p^{a-1})\right) \otimes \bar{C}\right)$$

of $H^0(G, R)$. The required factoring as a set map thus follows from the compatibility of the pairings and the fact that the lifted $c_a[i_a]^{(r-a)}$ are lifts of their mod p namesakes. □

Recall that G is the group scheme SL_2 over the noetherian base ring $\mathbf{k}$. Now let A be a finitely generated commutative $\mathbf{k}$-algebra with G action.

Theorem 2.12 *(CFG in rank one). $H^*(G, A)$ is a finitely generated algebra.*

Proof. Recall that A comes with an increasing filtration $A_{\leqslant 0} \subseteq A_{\leqslant 1} \subseteq \cdots$ where $A_{\leqslant i}$ denotes the largest G-submodule all whose weights λ satisfy $\operatorname{ht}\lambda = \sum_{\beta>0}\langle\lambda, \beta^\vee\rangle \leqslant i$. (Actually there is now only one positive root, so that the sum has just one term.) The associated graded algebra is the Grosshans graded ring $\operatorname{gr} A$. Let $\mathcal{A}$ be the Rees ring of the filtration. So $\mathcal{A}$ is the subring of the polynomial ring $A[t]$ generated by the subsets $t^i A_{\leqslant i}$. Let $\bar{A} = A/pA$. As in [5, Section 3] we choose r so big that $x^{p^r} \in \operatorname{gr}\bar{A}$ for all $x \in \operatorname{hull}_\nabla(\operatorname{gr}\bar{A})$. Put $\bar{C} = (\operatorname{gr}\bar{A})^{\bar{G}_r}$. By [2, Theorem 30] the algebra $\mathcal{A}/t\mathcal{A} = \operatorname{gr} A$ is finitely generated. Now t has degree one in the positively graded algebra $\mathcal{A}$, so $\mathcal{A}$ is also finitely generated. By [2, Theorem 35] the map $\operatorname{gr} A \to \operatorname{gr}\bar{A}$ is power surjective. Then so is the map $\mathcal{A} \to \operatorname{gr}\bar{A}$, because $\mathcal{A} \to \operatorname{gr} A$ is surjective. Now take a finitely generated G invariant subalgebra $\mathcal{C}$ of the inverse image of $\bar{C}$ in $\mathcal{A}$ in such a way that $\mathcal{C} \to \bar{C}$ is power surjective. By Theorem 2.11 the map $H^{\mathrm{even}}(G, \mathcal{C}) \to H^0(G, H^*(\bar{G}_r, \bar{C}))$ is noetherian. By [1, Theorem 1.5, Remark 1.5.1] the $H^*(\bar{G}_r, \bar{C})$-module $H^*(\bar{G}_r, \operatorname{gr}\bar{A})$ is noetherian and by [2, Theorems 9, 12] it follows that $H^0(G, H^*(\bar{G}_r, \bar{C})) \to H^0(G, H^*(\bar{G}_r, \operatorname{gr}\bar{A}))$ is noetherian. Then so is $H^{\mathrm{even}}(G, \mathcal{C}) \to H^0(G, H^*(\bar{G}_r, \operatorname{gr}\bar{A}))$, hence also $H^{\mathrm{even}}(G, \mathcal{A}) \to H^0(G, H^*(\bar{G}_r, \operatorname{gr}\bar{A}))$. This is what is needed to argue as in [5, 4.10] that $H^{\mathrm{even}}(G, \mathcal{A}) \to H^*(G, \operatorname{gr}\bar{A})$ is noetherian. And then one concludes as in [5, 4.11] that $H^{\mathrm{even}}(G, \mathcal{A}) \to H^*(G, \bar{A})$ is noetherian. But $\mathcal{A} \to \bar{A}$ factors through A. It follows that $H^{\mathrm{even}}(G, A) \to H^*(G, \bar{A})$ is noetherian. As p was an arbitrary prime, [2, Theorem 49], or rather the last item of [8, Theorem 10.5], applies. □

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