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# Constant terms in powers of a Laurent polynomial

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ABSTRACT

We classify the complex Laurent polynomials with the property that their powers have no constant term. The result confirms a conjecture of Mathieu for the case of tori. (A different case would imply Keller's Jacobian Conjecture.)

## 1. INTRODUCTION

We prove a special case of a conjecture of Mathieu ([Mat]).

**Conjecture 1 (Mathieu).** *Let $K$ be a connected real compact Lie group. Let $f$ and $g$ be $K$-finite functions on $K$. Assume that for all $n \geq 1$ the constant term of $f^n$ vanishes. Then for all but finitely many $n$ the constant term of $f^n g$ also vanishes.*

Here the constant term $\mathrm{Cst}(f)$ of $f$ is defined as the average

$$\int_K f(k)dk$$

of $f$ over $K$, the integral of $f$ with respect to the Haar measure, normalized so that $\int_K dk = 1$. In the canonical decomposition of $\mathbb{C}[K]$ in matrix coefficients of irreducible finite-dimensional representations of $K$, the constant term is given by the number $\mathrm{Cst}(f)$, which explains the name.

In this paper we prove the conjecture for commutative $K$. Therefore, from now on $K$ is a real torus and its complexification is an algebraic torus $T$ of rank $l$.

The ring of $K$-finite functions is the affine coordinate ring $\mathbb{C}[T]$ of $T$. The choice of a $\mathbb{Z}$-basis $z_1, \ldots, z_l$ in the character group $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, where $\mathbb{G}_m$ is the multiplicative group of the nonzero complex numbers, leads to an identification of $T$ with ${\mathbb{G}_m}^l$ and of $\mathbb{Z}^l$ with $X^*(T)$, under which $p \in \mathbb{Z}^l$ corresponds to the character (Laurent monomial) $z^p := \prod_{i=1}^{l} {z_i}^{p_i}$. A $K$-finite function $h$ then is nothing else than a Laurent polynomial in the $z_i$. And $\mathrm{Cst}(h)$ is just the constant term of the Laurent polynomial $h$. Or, if one thinks of $\mathbb{C}[T]$ as the linear span of the characters, then $\mathrm{Cst}(h)$ is the term corresponding with the trivial character. We will actually prove that *if* $\mathrm{Cst}(f^n) = 0$ *for all* $n \geq 1$, *then the trivial character does not belong to the convex hull of the characters which occur in* $f$ *with nonzero coefficients*, where we view $X^*(T) \simeq \mathbb{Z}^l$ as a lattice in $X^*(T) \otimes \mathbb{R} \simeq \mathbb{R}^l$.

## 2. ONE VARIABLE

We start with the case $l = 1$, where $T = \mathbb{G}_m$ and $K$ is the circle, as it is much more elementary and yet illustrates the method.

**Theorem 2.** *Assume that* $f \in \mathbb{C}[z, z^{-1}]$ *is neither a polynomial in* $z$ *nor a polynomial in* $z^{-1}$. *Then* $f$ *has a critical value* $v \in \mathbb{C} \setminus \{0\}$, *such that* $\limsup_{n \to \infty} |\mathrm{Cst}(f^n)|^{1/n} = |v| > 0$.

**Proof.** We have $\mathbb{C}[z, z^{-1}] = \mathbb{C}[C]$ where $C$ is the unit circle in $\mathbb{C} \setminus \{0\}$. We consider the generating function

$$F(t) := \sum_{n=1}^{\infty} \mathrm{Cst}(f^n) t^{n-1} = \frac{1}{2\pi i} \int_C \frac{f(z)}{1 - tf(z)} \frac{dz}{z},$$

where we have used that averaging over $C$ is equal to the complex line integral over $C$ with respect to $\frac{dz}{2\pi i z}$. For small $|t|$ this defines a holomorphic function of $t$ which is equal to the sum of the residues of the function $f(z)/(1 - tf(z))z$ at its poles $z$ with $|z| < 1$ (or minus the sum of the residues for $|z| > 1$). Because $f(0) = \infty$, the residue at $z = 0$ is equal to $-1/t$. As long as $\tau = 1/t$ is not a critical value of $f$ and $|t|$ is small, the other residues are equal to $-1/(t^2 f'(\zeta)\zeta)$, where $\zeta = \zeta_j(\tau)$ ranges over the solutions of $f(\zeta) = \tau$ such that $|\zeta| < 1$.

Along every curve in $\mathbb{C}$ which avoids the critical values of $f$, the functions $\tau \mapsto \zeta_j(\tau)$ have a complex analytic extension. The idea of the proof is to show that the asymptotic behavior for $t \to \infty$ of the corresponding analytic continuation of $t \mapsto F(t)$ will lead to the conclusion that $F(t)$ is not identically zero, and actually has a finite radius of convergence.

Because $f(0) = f(\infty) = \infty$, the complex analytic extension of $\tau \mapsto \zeta_j(\tau)$ can neither run to 0, nor to $\infty$, when $\tau$ remains bounded. If $a \in \mathbb{C}$ is a critical point of $f$, with corresponding critical value $v$, then there exists an integer $m \geq 2$ and a nonzero complex number $c$, such that $f(z) \sim v + c(z - a)^m$ and $f'(z) \sim cm(z - a)^{m-1}$ as $z \to a$. It follows that the solutions $\zeta$ near $a$ of $f(\zeta) = \tau$ satisfy $\zeta - a \sim (\frac{\tau - v}{c})^{1/m}$, with a choice of branch of the $m$-th root. We get that for $v \neq 0$ the residue at $\zeta$ is of the order $(\tau - v)^{-1+1/m}$.

For $v = 0$ the residue at $\zeta$ is of order $\tau^{1+1/m}$, which cannot cancel the residue $-\tau$ at $z = 0$. (Take $m = 1$ if $\zeta$ approaches a simple zero $a$ of $f$.) The conclusion is that $F(t) = -t^{-1} + \mathcal{O}(t^{-1-1/m})$ for some $m \geq 1$ as $t \to \infty$, which shows that $F(t)$ is not identically equal to zero. Even stronger, if around the nonzero critical value $v$ of $f$ the complex analytic extension of $F$ would be single-valued, then the estimate $|F(t)| \leq C\,|t - \frac{1}{v}|^{-1+1/m}$ for $t$ near $1/v$ in combination with Cauchy's integral formula shows that $F$ has a holomorphic extension to a neighborhood of $1/v$. If this holds for every nonzero critical value $v$ of $f$, then $F$ extends to an entire analytic function on $\mathbb{C}$, such that $F(t) = -t^{-1} + \mathcal{O}(t^{-1-1/m}) \to 0$ as $t \to \infty$. By Liouville's theorem we would arrive at the conclusion that $F(t) \equiv 0$, a contradiction. In particular we get that the radius of convergence of the generating function $F(t)$ is equal to $|1/v|$, where $v$ is a nonzero critical value of $f$, which implies the statement of the theorem.

**Remark 3.** If for every $n \geq 1$ the constant term of $f^n$ vanishes, then Theorem 2 implies that either $f \in \mathbb{C}[z]$ or $f \in \mathbb{C}[z^{-1}]$, without constant term, and we get the conclusion of Mathieu's conjecture for the circle.

In the proof we actually showed that if neither $f \in \mathbb{C}[z]$ nor $f \in \mathbb{C}[z^{-1}]$, then there exists a nonzero critical value $v$ of $f$, such that the radius of convergence of the generating function $F$ is equal to $|1/v|$, and the complex analytic extension around $1/v$ of $F(t)$, $|t| < |1/v|$, is not single-valued.

## 3. ARBITARY RANK

Recall that the Newton polytope Newton$(h)$ of a $K$-finite function $h$ is the convex hull in the vector space $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ of the characters that occur in $h$. (Or, with coordinates given, the convex hull in $\mathbb{R}^l$ of the multi-indices that occur as exponents in the Laurent polynomial.) We have two cases to consider. The first is that the $f$ in the conjecture is such that the origin $\mathcal{O}$ is outside Newton$(f)$. That case is easy. As the Newton polytope is a finite intersection of rational half spaces (cf. [O]), there is a cocharacter $y \in X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) = \mathrm{Hom}(X^*(T), \mathbb{Z})$ taking positive values on Newton$(f)$. Therefore there is a basis of $X_*(T)$ so that in the corresponding coordinates the first variable $z_1$ occurs in $f$ with positive powers only. Then the conclusion in Mathieu's conjecture is visibly true for $f$.

The hard case is thus when $\mathcal{O} \in$ Newton$(f)$. In this case we may assume that $\mathcal{O}$ is actually in the interior of Newton$(f)$. For suppose $\mathcal{O}$ lies on the boundary. Then it lies in the interior of some face $F$, with respect to the topology of the smallest affine subspace $L$ which contains $F$. Because $\mathcal{O} \in F$, $L$ is a vector subspace. Let $\tilde{f}$ be the sum of the terms in $f$ that are in the span $\mathbb{C}[L \cap X^*(T)]$ of $L \cap X^*(T)$. This is a Laurent polynomial in fewer variables, for which we can take a $\mathbb{Z}$-basis of $L \cap X^*(T)$. As the rest of the Newton polytope lies on one side of $L$, we have $\mathrm{Cst}(f^n) = \mathrm{Cst}(\tilde{f}^n)$. We may replace $f$ with $\tilde{f}$ and $T$ with the torus $\tilde{T}$ whose character group is $L \cap X^*(T)$. After these replacements the origin lies in the interior of the Newton polytope. The reader will have noticed that $L$ may have dimension zero.

In the sequel we will make use of the 'Haar form' $\omega := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_l}{z_l}$, which modulo a constant factor is the unique invariant $(l, 0)$-form on $T$.

**Theorem 4.** *Let $f \in \mathbb{C}[T]$ be such that the origin $\mathcal{O}$ lies in the interior of the Newton polytope* Newton$(f)$. *Then there exists a smooth compactification $M$ of $T$ such that $f$ extends to a holomorphic mapping from $M$ to $\mathbb{P}^1$, and the Haar form $\omega$ extends to a holomorphic differential form on $M \setminus f^{-1}(\{\infty\})$.*

**Theorem 5.** *If $M$ is as in the conclusion of Theorem 4, then there exists a nonzero critical value $v$ of the mapping $f : M \to \mathbb{P}^1$, such that the radius of convergence of the generating function $F(t) = \sum_{n=1}^{\infty} \mathrm{Cst}(f^n) t^{n-1}$ is equal to $1/|v|$ and the complex analytic extension around $1/v$ of $F(t)$, $|t| < |1/v|$, is not single-valued. In particular,* $\limsup_{n\to\infty} |\mathrm{Cst}(f^n)|^{1/n} = |v| > 0$.

The proof of the theorems follows the same line as its special case Theorem 2. When $l = 1$, we had no difficulty extending $f : \mathbb{C} \to \mathbb{C}$ to a map $f : \mathbb{P}^1 \to \mathbb{P}^1$. We did not even mention it. We did use though that $f$ had limit $\infty$ at zero and infinity, which were the poles of $dz/z$. For general $l$ a compactification with similar properties exists, but the proof requires toroidal compactification ([KKMD],[O]) and Hironaka's resolution of singularities [H].

**Proof of Theorem 4.** We start by constructing an $M$ with fewer properties and then improve on it. As our first attempt we take the toroidal compactification $M_{\mathrm{tor}}$ associated with a fan $F$ which is a finite nonsingular subdivision of the fan consisting of the cones on the proper faces of the polar polytope of Newton$(f)$ in the sense of Oda [O]. Every $l$-dimensional cone $\sigma$ in $F$ is spanned by a $\mathbb{Z}$-basis of $X_*(T)$. We use its dual basis $z_i$, $1 \le i \le l$, as a coordinate system on $T$. Extending $\mathbb{G}_m{}^l$ to $\mathbb{C}^l$ we get the chart $Y_\sigma$ of $M_{\mathrm{tor}}$ corresponding to $\sigma$. There is a unique vertex $m$ of Newton$(f) \subset X^*(T) \simeq \mathbb{Z}^l$ such that for each $p \in$ Newton$(f)$ and $1 \le i \le l$ we have $p_i \ge m_i$. It follows that $f(z) = \phi(z)\, z^m$ for a polynomial $\phi(z)$, such that $\phi(0) \ne 0$. Moreover, the condition that $\mathcal{O}$ is in the interior of Newton$(f)$ implies that, for each $i$, $m_i < 0$. Therefore $f$ is well-defined and equal to infinity at $z = 0$.

Also, $\omega = \pm \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_l}{z_l}$, so in these coordinates $\omega$ again has simple poles and pole $(\omega)$, the divisor of the poles of $\omega$, consists of the coordinate hyperplanes of the chart $Y_\sigma$. We stratify pole$(\omega)$ by repeated intersection of its irreducible components. We see that $f : M_{\mathrm{tor}} \to \mathbb{P}^1$ is well defined, with the value $\infty$, on a dense open subset of each stratum of pole$(\omega)$, because it has this property near the zero-dimensional stratum, which is the origin in our chart. Now $M_{\mathrm{tor}}$ is the union of the charts $Y_\sigma$, so we may make the same remarks for the full divisor of poles of $\omega$ on $M_{\mathrm{tor}}$, which we call pole$(\omega)$ again.

So the thing that is still missing is that $f$ be defined everywhere. Actually, if $l > 1$ then for every finite value of $\tau$ the level set $f^{-1}(\{\tau\})$ in $T$ cannot be a compact subset of $T$, because of the maximum principle, applied to the restriction to the level set of the coordinate functions $z_i$. At a limit point in $M_{\text{tor}} \setminus T$ of the level set we get also $f = \infty$ at arbitrarily nearby points, so $f$ must have points where it is ambiguous.

If we can make $f$ defined everywhere, without spoiling the property that generically on each stratum of pole($\omega$) the value of $f$ is $\infty$, then Theorem 4 follows. Now we are in luck. Hironaka tells us in [H, §5 of Ch. 0] how to make $f$ well defined everywhere by performing a finite sequence of monoidal transformations (also known as blowups [BM, §2]) with smooth centers. At each stage the center of his blowup is contained in the locus where $f$ is still ambiguous. (This locus may be described as the scheme theoretic intersection of the divisor of poles of $f$ with the divisor of zeroes of $f$. Thus, initially its ideal sheaf $J$ is locally, in the chart $Y_\sigma$, the ideal generated by the polynomials $\phi$ and $z^{-m}$ for which $f(z) = \phi(z)\, z^m$.) Moreover, he appeals to his Main Theorem II and that means we may, apart from $J$, also specify a divisor $E_0$ which has only normal crossings, for which we take pole($\omega$), of course.

At the $i$-th stage the center will have only normal crossings with a divisor $E_i$, inductively defined in the Main Theorem II. Let $\omega_i$ and $f_i$ denote $\omega$ and $f$ at the $i$-th stage, respectively. Using computations in local coordinates as in [BM, §2] or [GH, p. 603], one then gets by induction on $i$:

(A) $\omega_i$ is a meromorphic differential form with at most simple poles, along a divisor pole($\omega_i$) with normal crossings. Each component of pole($\omega_i$) is equal to a component of $E_i$.

(B) Each stratum $S$ of pole($\omega_i$) is equal to the closure of the complement in $S$ of the exceptional divisor and therefore, by induction on $i$, $f_i = \infty$ on an open dense subset of $S$.

From (A) one gets that the center at the $i$-th stage has only normal crossings with pole($\omega_i$), and from (B) one knows it does not contain any stratum $S$. (Recall the center is contained in the locus of ambiguity of $f_i$. The terminology 'normal crossings' would still allow the center to contain a stratum, but we do not want this.) This is then used in the induction step. Note that the complement of the center of any monoidal transformation is identified with the complement in the blowup of the exceptional divisor [GH, p. 604]. Property (B) means that pole($\omega_i$) is equal to the so-called strict transform of pole($\omega_{i-1}$).

At the end of the sequence of blowups we have a smooth compactification $M$ of $T$ such that $f$ extends to a holomorphic mapping $f : M \to \mathbb{P}^1$. Moreover, $f = \infty$ on a dense open subset of each stratum of pole($\omega$), so by continuity $f = \infty$ on pole($\omega$).

**Remark 6.** If $\mathcal{O}$ is outside Newton($f$) then the locus of ambiguity of $f$ will contain some stratum of pole($\omega$). Actually, in this case the conclusion of Theorem 5 fails. Hence the conclusion of Theorem 4 cannot hold if $\mathcal{O} \notin$ Newton($f$).

**Remark 7.** There is another kind of blowup which also does not spoil the property that generically on each stratum of pole($\omega$) the value of $f$ is $\infty$. Namely the blowup of the closure of a stratum. (Of course in this case the number of strata of pole($\omega$) does increase.) Using [Mo, Prop. 6.5] we can replace our $M_{\text{tor}}$ by a toroidal compactification of $T$ which also is a projective variety. Because the blowup of a projective variety is projective, we may therefore modify the proof to achieve that $M$ in Theorem 4 is projective.

## 4. RESIDUES

We again study the generating function

$$F(t) := \sum_{n=1}^{\infty} \mathrm{Cst}(f^n)t^{n-1} = (2\pi i)^{-l}\int_K \frac{f(z)\omega}{1 - tf(z)},$$

where $K$ is the real torus $|z_i| = 1$, $1 \leq i \leq l$. For this purpose we return to a coordinate chart $Y_\sigma \cong \mathbb{C}^n$ of the toroidal compactification $M_{\text{tor}}$ from the beginning of the proof of Theorem 4. Thus we work with a coordinate system $z_1, \ldots, z_l$ on $T$ so that there is a term $m$ in $f$ whose $z_i$-degree is, for each $i$, strictly negative and less than or equal to the $z_i$-degree of each term of $f$. The derivative of $f$ with respect to $z_1$ will similarly have such a lowest monomial, so there is a neighborhood $N$ of the origin where both $f$ and this derivative are defined and nonzero. (Infinite value is of course allowed here.) Note that for nonzero $t$ the form $\frac{\omega}{f(z)-(1/t)}$ has no pole along $z_1 = 0$ in $N$. Take $\epsilon > 0$ so that $(z_1, \ldots, z_l)$ lies in $N$ when all $|z_i|$ are no larger than $\epsilon$. Let $S_\epsilon$ denote the circle of radius $\epsilon$ and center 0 in $\mathbb{C}$ and let $D_\epsilon$ be the disc it bounds. Let $K_\epsilon$ be the real $l$-dimensional cycle $(S_\epsilon)^l$ in $T$. For small $|t|$ we may replace $K$ by $K_\epsilon$. Putting

$$J(\tau) := \frac{1}{2\pi i}\int_{K_\epsilon} \frac{\omega}{f(z) - \tau},$$

we get that

$$F(t) = -\frac{1}{t} - \frac{1}{t^2}(2\pi i)^{1-l}J(1/t)$$

for small $|t|$. We will investigate the analytic continuation of $J(\tau)$, initially defined for large $|\tau|$, in particular for $\tau \to 0$.

As in the one-dimensional case, we first rewrite $J(\tau)$ in terms of residues. The $(l+1)$-dimensional cycle $D_\epsilon \times (S_\epsilon)^{l-1}$ has $K_\epsilon$ as its boundary and it intersects the pole divisor of $\frac{\omega}{f-\tau}$ transversally in $\sigma_\tau = f^{-1}(\{\tau\}) \cap (D_\epsilon \times (S_\epsilon)^{l-1})$. Applying Cauchy's integral formula to the integration over $z_1$ we get the theorem of Leray ([L], see also [BGVY, §16 of Ch. 3]), stating that

$$J(\tau) = \int_{\sigma_\tau} \omega/df,$$

where we use the suggestive notation $\omega/df$ for the Poincaré residue of $\frac{\omega}{f(z)-\tau}$.

In the complement of the set of critical points of $f$, $f^{-1}(\{\tau\})$ is a smooth complex hypersurface, on which $\omega/df$ is the holomorphic $(l-1,0)$-form $\eta$, which is determined by the following property. If $v_2, \ldots, v_l$ are tangent vectors to $f^{-1}(\{\tau\})$ and if $v_1$ is a tangent vector to $T$ at the same point such that $\langle v_1, df\rangle = 1$, then $\eta(v_2, \ldots, v_n) = \omega(v_1, v_2, \ldots, v_n)$. The 'relative $(l-1,0)$-form' $\omega/df$ can also be described as the restriction to $f^{-1}(\{\tau\})$ of any smooth $(l-1,0)$-form $\mu$ in an open subset of $M$, such that $\omega = df \wedge \mu$. Locally in the regular set of $f$, $\mu$ can be chosen to be holomorphic.

For integrals $J(\tau)$, of relative holomorphic (algebraic) $(l-1,0)$-forms over real $(l-1)$-dimensional cycles $\sigma_\tau$ in $f^{-1}(\{\tau\})$, where the cycles $\sigma_\tau$ move continuously with $\tau$, and initially defined for $\tau$ in a neighborhood of a given point in $\mathbb{C}$, the following general facts are known.

(i) The function $J$ is of Nilsson class, which means that there is a finite subset $V$ of $\mathbb{C}$ such that $J$ has a complex analytic extension along every curve in $\mathbb{C}\setminus V$. And moreover, in each sector near each $v \in V$, the extension can be written as a finite sum

$$J(\tau) = \sum_{\alpha,q} C^v_{\alpha,q}(\tau)(\tau - v)^\alpha(\log(\tau - v))^q,$$

where $\alpha \in \mathbb{C}$, $q \in \mathbb{Z}_{\geq 0}$, $C^v_{\alpha,q}$ is holomorphic in a neighborhood of $v$ and $C^v_{\alpha,q}(v) \neq 0$.

(ii) All exponents $\alpha$ in (i) are rational numbers. (In the case of isolated singularities this is Brieskorn's monodromy theorem.)

(iii) All exponents $q$ in (i) satisfy $q \leq l-1$.

Property (i) was proved by Nilsson [N1]. Using a semi-algebraic triangulation of our cycle $\sigma_\tau$ for large $|\tau|$, we can write our initial $J(\tau)$ as a finite sum of integrals over the standard $(l-1)$-dimensional simplex of an algebraic function, so we can get (ii) and (iii) by applying Nilsson [N2]. Griffiths, Katz and Deligne showed that the use of compactifications and Hironaka's resolution of singularities (which were not used by Nilsson) leads to simpler proofs in a more general framework, cf. [D, prop. 6.14 and Th. 7.9 in Ch. II, Th. 1.8 and Th. 2.3 in Ch. III].

These general results do not give much information on the set $V$ of singular points for $J$, nor on the exponents $\alpha$ which may occur. We will show in the next section that if $M$ is as in Theorem 4 then $V$ is contained in the set of critical values of $f: M \to \mathbb{P}^1$. More importantly, for each $v \in V$ and exponent $\alpha$ occurring in the description of $J$ near $v$, we have that $\alpha > -1$. As in the case $l = 1$, this then will lead to a proof of Theorem 5.

## 5. ASYMPTOTICS

In this section we assume throughout that $M$ is as in the conclusion of Theorem 4. We write $M_\tau$ for the level set in $M$ at the level $\tau \in \mathbb{P}^1$, for the holomorphic mapping $f: M \to \mathbb{P}^1$. If $\tau \in \mathbb{C}$ is not a critical value of $f$, then $M_\tau$ is a complex analytic smooth hypersurface in $M$, on which we have a well-defined holomorphic $(l-1,0)$-form $\omega/df$.

Because $f$ is constant on each of the finitely many strata of the variety of critical points of $f$, the set $V$ of critical values of $f$ is finite. (This could be called Sard's Theorem for proper holomorphic maps.) Then over $U := \mathbb{C}/V$ one has a locally trivial $C^\infty$ fibration, hence a Gauss-Manin connection on the sheaf of $(l-1)$-dimensional (co)homology groups of the fibers $M_\tau$. (Compare the discussion in [A, §3 of Ch. 2] in the case where $f$ has an isolated singularity.) To extend $J(\tau)$ holomorphically along a path $\gamma$ in $U$ one must simply vary the cycle $\sigma_\tau \in M_\tau$ continuously with $\tau$. (This follows from the theorem of Leray, which allows us to go back and forth between an integral in the fiber $M_\tau$ and an integral of $\frac{\omega}{f(z)-\tau}$ over a cycle in the boundary of a tubular neighborhood of $M_\tau$.)

Let us choose a smooth lift $\nabla$ in the tangent bundle of $f^{-1}(U)$ of the vector field $\frac{d}{dz}$ on $U$. Say by choosing a hermitian metric on the tangent bundle of $M$ and then taking $\nabla$ at each point to be the appropriate complex multiple of the gradient of $f$. For every path $\gamma$ in $U$ which starts near $\infty$, the transport of $\sigma_\tau$ parallel to the path $\gamma$ with respect to the connection $\nabla$ will give an analytic continuation of $J(\tau)$ along $\gamma$.

**Lemma 8.** *For each exponent $\alpha$ in the asymptotic expansion of $J$ near a singular point, we have $\alpha > -1$.*

Such inequalities are proved in [J] and [Mal2] and, under a different guise, in [K]. Neither [J] nor [Mal2] applies exactly in our setting. Jeanquartier studies a real analytic function, so his fibers have real codimension one, and Malgrange works near an isolated singularity of $f$. Our proof follows [Mal1, Appendice], the statements in which are not directly applicable to our situation either.

**Proof of Lemma 8.** We have to estimate the growth of $J(\tau)$ as we approach a critical value $v$ along a ray. By passing to $f - v$, we can arrange that $v = 0$, and for the ray we can take the positive real axis, which simplifies the notation.

Let $\tau_0 > 0$ be such that $[0, \tau_0] \subset U$ and observe that there is a deformation retraction of $f^{-1}([0, \tau_0])$ onto $M_0$, say by [Lo1]. Therefore $\sigma_{\tau_0}$ is homologous in $f^{-1}([0, \tau_0])$ to a cycle in $M_0$. By Łojasiewicz [Lo2] one may put a semi-analytic triangulation on $f^{-1}([0, \tau_0])$, or rather on the pair $(f^{-1}([0, \tau_0]), M_0 \cup M_{\tau_0})$. Then $\sigma_{\tau_0}$ is homologous in $M_{\tau_0}$ to a cycle $\Gamma_{\tau_0}$ in the triangulation. As we have seen that it is homologous in $f^{-1}([0, \tau_0])$ to a cycle in $M_0$, there is a chain $\Delta$ in the triangulation with $\partial\Delta = \Gamma_{\tau_0} - \Gamma_0$, where $\Gamma_0 \subset M_0$. By Herrera [He] we can integrate over semi-analytic chains and this has the usual properties with respect to homology. And one has a Stokes' Theorem.

The 'chain' $\Delta_\tau = f^{-1}([0, \tau]) \cap \Delta$ is semi-analytic for $\tau \in [0, \tau_0]$. To make it into a true chain in the triangulation one must subdivide the triangulation, using [Lo2] again, so that $M_\tau = f^{-1}(\tau)$ is a subcomplex. Then $\partial\Delta_\tau = \Gamma_\tau - \Gamma_0$ with $\Gamma_\tau \subset M_\tau$. And $\partial(\Delta - \Delta_\tau) = \Gamma_{\tau_0} - \Gamma_\tau$, so $\Gamma_\tau$ represents the same homology class as $\Gamma_{\tau_0}$ or $\sigma_{\tau_0}$ in $f^{-1}([\tau, \tau_0]) \simeq (M_{\tau_0} \times [\tau, \tau_0])$. Thus

$$J(\tau) = \int_{\Gamma_\tau} \omega/df.$$

Write

$$\Delta(\tau, \tau') := \Delta \cap f^{-1}([\tau, \tau']),$$

$$I(\tau) := \int_{\Delta(0,\tau)} \omega.$$

Then $I(\tau)$ is bounded on $[0, \tau_0]$ because the semi-analytic chain $\Delta$ has finite $l$-dimensional Euclidean volume, cf. [He, II.A.2.1(c) and I.C.1.]. By Lemma 9 below, $J(\tau)$ is the derivative with respect to $\tau$ of the bounded function $I(\tau)$. Then the leading term in the asymptotic expansion of $J(\tau)$ must have exponent $\alpha > -1$, as claimed. It thus remains to prove:

**Lemma 9.** $J(\tau) = I'(\tau)$ *for* $\tau \in ]0, \tau_0[$.

**Proof.** In the (open) complement of the set of zeros of $df$ in $M$ there exists a smooth $(l-1, 0)$-form $\mu$ such that $\omega = df \wedge \mu$. This is obvious locally and the global statement follows by means of a smooth partition of unity. Let $\tau, \tau' \in ]0, \tau_0[$. Integrating $\omega = d((f-\tau)\mu) + (\tau - f)d\mu$ over $\Delta(\tau, \tau')$, and applying the formula of Stokes in the version of Herrera [He] to the integral of the first term in the right hand side, we get $I(\tau') - I(\tau) = (\tau' - \tau)J(\tau') + \int_{\Delta(\tau,\tau')}(\tau - f)d\mu$. Dividing by $\tau' - \tau$ and using that the function $J$ is continuous, $\left| \frac{\tau - f}{\tau' - \tau} \right| \leq 1$ in $\Delta(\tau, \tau')$ and the $l$-dimensional Euclidean volume of $\Delta(\tau, \tau')$ converges to zero as $\tau' \to \tau$, cf. [He, II.A.2.1(c) and I.C.1], we get that $\lim_{\tau' \to \tau} \frac{I(\tau') - I(\tau)}{\tau' - \tau} = J(\tau)$.

**Remark 10.** It follows that $I(\tau_0) = \int_{[0,\tau_0]} J(\tau)d\tau$, which is a Fubini-type of formula 'the integral is equal to the integral over the base of the integral over the fiber' for the fibration $f$; note that $(\omega/df)d\tau = \omega$ when $\tau = f$.

As in [Mal2, p. 13], the estimates along rays lead to an independent proof that $J$ is of Nilsson class, with the additional property that for every exponent $\alpha$ we have Re $\alpha > -1$. This is sufficient for the proof of Theorem 5.

**Remark 11.** In the proof of lemmas 8 and 9 we did not need to assume that $M_0$ has normal crossings. Let us assume that now. One may think of $f\omega/df$ as a holomorphic section of the sheaf of relative differential $(l-1)$-forms with logarithmic poles along $(M_0)_{\text{red}}$. This section vanishes along $(M_0)_{\text{red}}$. By applying Mumford's Semi-stable Reduction Theorem [KKMD, Ch. 2] one could further arrange that $M_0$ is a reduced divisor with normal crossings, so that the section $\omega/df$ also extends over $M_0$. In any case, one may now check that the smooth representative $f\mu$ of $f\omega/df$ can be chosen to extend over $M_0$ such that its restriction to $M_0$ vanishes. That may be used to give a proof of Lemma 8 that is even closer to [Mal1, Appendice]. Alternatively, if one seeks a more algebraic geometric proof of Lemma 8, one may combine the above extendability of $\omega/df$ with the monodromy theorem of Katz [K].

**Proof of Theorem 5.** Using the fact that $J$ is of Nilsson class with exponents $\alpha$ such that Re $\alpha > -1$, the proof proceeds as in the case $l = 1$. Indeed, it then follows that $J$ can be holomorphically extended to a neighborhood of every critical value around which $J$ is single-valued. If this happens for all nonzero critical values of $f$, then $J$ is a single-valued holomorphic function in $\mathbb{C} \setminus \{0\}$, so extends to an entire analytic function on $\mathbb{C}$. Considering the asymptotic behavior of $F(t) = -\frac{1}{t} - \frac{1}{t^2}(2\pi i)^{1-l} J(1/t)$ near $t = 0$, we then see that $F$ is a nonzero entire analytic function on $\mathbb{C}$, which moreover converges to 0 when $t \to \infty$, in contradiction with Liouville's theorem.

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## REFERENCES

[A] Arnol'd, V.I. (Ed.) – Dynamical Systems VI. Encyclopaedia of Mathematical Sciences **6**. Springer-Verlag, Berlin (1993).

[BGVY] Berenstein, C.A. et al. – Residue Currents and Bezout Identities. Progress in Mathematics **114**. Birkhäuser, Basel (1993).

[BM] Bierstone, E. and P.D. Milman – A simple constructive proof of canonical resolution of singularities. p. 11–30 in: Effective Methods in Algebraic Geometry (Castiglioncello 1990), Progress in Mathematics **94**. Birkhäuser, Boston (1991).

[D] Deligne, P. – Équations différentielles à points singuliers réguliers. Lect. Notes in Math. **163**. Springer-Verlag, Berlin (1970).

[GH] Griffiths, P. and J. Harris – Principles of Algebraic Geometry. Wiley & Sons, New York (1978).

[H] Hironaka, H. – Resolution of singularities of an algebraic variety over a field of characteristic zero I. Annals of Math. **79**, 109–203 (1964).

[He] Herrera, M. – Integration on a semi-analytic set. Bull. Soc. Math. Fr. **94**, 141–180 (1966).

[J] Jeanquartier, P. – Dévelopement asymptotique de la distribution de Dirac. C. R. Acad. Sc. Paris, t. **271**, série A, 1159–1161 (1970).

[K] Katz, N. – The regularity theorem in algebraic geometry. p. 437–443 in: Actes du congrès international des mathématiciens, 1970, **1**. Gauthiers-Villars, Paris (1971).

[KKMD] Kempf, G. et al. – Toroidal Embeddings I. Lect. Notes in Math. **339**. Springer-Verlag, Berlin (1973).

[L] Leray, J. – Le calcul différentiel et intégral sur une variété analytique complexe (Problème de Cauchy III). Bull. Soc. Math. Fr. **87**, 81–180 (1959).

[Lo1] Łojasiewicz, S. – Une propriété topologique des sous-ensembles analytiques réels. p. 87–89 in: Les Équations aux Dérivées Partielles, Coll. Int. CNRS **117** (1962). Éditions CNRS, Paris (1963).

[Lo2] Łojasiewicz, S. – Triangulations of semi-analytic sets. Ann. Scuola Norm. Sup. Pisa, III **18**, 449–474 (1964).

[Mal1] Malgrange, B. – Sur les polynômes de I.N. Bernstein. Exposé no. 20 in Séminaire Goulaouic-Schwartz 1973-74. École Polytechnique, Paris (1974).

[Mal2] Malgrange, B. – Intégrales asymptotiques et monodromie. Ann. scient. Éc. Norm. Sup. 4° série 7, 405–430 (1974).

[Mat] Mathieu, O. – Some conjectures about invariant theory and their applications. preprint, Strasbourg 1996, to appear in: J. Alev (ed.), Proceedings of the Reims conference, 1995, Séminaires et Colloques de la Soc. Math. France.

[Mo] Morelli, R. – The birational geometry of toric varieties. J. Algebraic Geometry **5**, 751–782 (1996).

[N1] Nilsson, N. – Some growth and ramification properties of certain integrals on algebraic manifolds. Arkiv för Matematik **5**, 463–476 (1964).

[N2] Nilsson, N. – Monodromy and asymptotic properties of certain multiple integrals. Arkiv för Matematik **18**, 181–198 (1980).

[O] Oda, T. – Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. **15**. Springer-Verlag, Berlin (1988).

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