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A PRESENTATION FOR SOME $K_2(n, R)$

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1. All rings are commutative with identity. We announce a presentation for the K_2 of a class of rings which includes the local ones. We also give a presentation for the relative K_2 of a homomorphism that splits and has its kernel in the Jacobson radical. These results generalize (and were suggested by) various earlier ones: the presentation of Matsumoto for the K_2 of (infinite) fields [6], [7, §11, 12]; the presentation of Dennis and Stein for the K_2 of discrete valuation rings and homomorphic images thereof [2]; stability results of the same authors [4]; the presentation for the relative K_2 of dual numbers, by one of us [5]. We reproved most of the earlier results and generalized them in the process.

2. **The functor** D (cf. [3, §9]).

2.1. Let R be a ring, R^* its group of units. We define the abelian group $D(R)$ by the following presentation:

Generators are the symbols $\langle a, b\rangle$ with $a, b \in R$ such that $1 + ab \in R^*$.

Relations are: (D0) commutativity.

(D1) $\langle a, b\rangle \langle -b, -a\rangle = 1$.

(D2) $\langle a, b\rangle \langle a, c\rangle = \langle a, b + c + abc\rangle$.

(D3) $\langle a, bc\rangle = \langle ab, c\rangle \langle ac, b\rangle$.

In all of these relations it is assumed that the left-hand sides make sense. For instance, in (D3) one needs $a, b, c \in R$ with $1 + abc \in R^*$. D is a functor from (commutative) rings to abelian groups. It commutes with finite direct products.

2.2. Put $K_2(n, R) = \ker(\mathrm{St}(n, R) \longrightarrow \mathrm{SL}(n, R))$, so that $K_2(R) = \varinjlim K_2(n, R)$. Put $K_2(\infty, R) = K_2(R)$. Relations (D1), (D2), (D3) imply the relations in [3, §9] and vice versa. So the rule

$$\langle a, b\rangle \longmapsto x_{21}\left(\frac{-b}{1+ab}\right)x_{12}(a)x_{21}(b)x_{12}\left(\frac{-a}{1+ab}\right)h_{12}^{-1}(1+ab)$$

defines a homomorphism $D(R) \longrightarrow K_2(R)$ factoring through $K_2(3, R)$.

2.3. DEFINITION. R is called *3-fold stable* if, for any triple of unimodular sequences $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ there exists $r \in R$ such that $a_i + b_i r \in R^*$ for $i = 1, 2, 3$. (Recall that (a, b) is called unimodular if $aR + bR = R$.) Similar definitions can be given for k-fold stability, e.g., 1-fold stability is the strongest of Bass' stable range conditions $SR_n(R)$ [1]. The condition of 3-fold stability is

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still stronger than that of 1-fold stability.

2.4. THEOREM 1. *Let R be local or 3-fold stable. Then* $D(R) \longrightarrow K_2(n, R)$ *is an isomorphism for* $3 \leqslant n \leqslant \infty$.

2.5. Now let I be an ideal contained in the Jacobson radical $\mathrm{Rad}(R)$ of R. The abelian group $D(R, I)$ is defined by the following presentation:

Generators are the $\langle a, b \rangle$ with $a \in R$, $b \in I$ or $a \in I$, $b \in R$.

Relations are: commutativity; (D1) for $a \in I$, $b \in R$; (D2) for $a \in R$, $b, c \in I$; (D2) and (D3) for $a \in I$, $b, c \in R$. (See 2.1 and compare [8, §2].) As in 2.2, one has a homomorphism $D(R, I) \longrightarrow K_2(R)$. It factors through $D(R)$.

2.6. THEOREM 2. *Let I be an ideal, contained in* $\mathrm{Rad}(R)$, *such that* $R \longrightarrow R/I$ *splits. Then*

$$1 \longrightarrow D(R, I) \longrightarrow K_2(n, R) \longrightarrow K_2(n, R/I) \longrightarrow 1$$

is split exact for $3 \leqslant n \leqslant \infty$.

2.7. THEOREM 3. *Let* $f: R \longrightarrow S$ *be a homomorphism of rings inducing an isomorphism* $R/\mathrm{Rad}(R) \longrightarrow S/\mathrm{Rad}(S)$. *If* $3 \leqslant n \leqslant \infty$ *and* $D(R) \longrightarrow K_2(n, R)$ *is an isomorphism, then* $D(S) \longrightarrow K_2(n, S)$ *is an isomorphism.*

2.8. EXAMPLES AND REMARKS. (1) A semilocal ring is k-fold stable if and only if all its residue fields contain at least $k + 1$ elements.

(2) The ring of continuous complex valued functions on a 1-dimensional complex is k-fold stable for any $k \in \mathbf{N}$.

(3) The ring of all totally real algebraic integers in $\mathbf{C}$ is k-fold stable for any $k \in \mathbf{N}$ (H. W. Lenstra).

(4) If R is 5-fold stable, then we can also show that $K_2(R)$ can be presented by Matsumoto's relations [3, §11]. For local rings with infinite residue fields the analogous result holds for any type of Chevalley group (cf. [6, Corollaire 5.11]).

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