1 Lie algebra cohomology

Lie algebra cohomology was invented by E.Cartan in an attempt to compute the de Rham cohomology of a compact Lie group. Thus let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$. The de Rham cohomology is computed by way of a complex Ω^*G of smooth differential forms on G. By averaging one may deform an arbitrary form on G to a right invariant one. One easily shows (see Bourbaki) that that the averaging process induces a map $\Omega^*G \to (\Omega^*G)^G$ which is homotopic to the identity map, so that de Rham cohomology may just as well be computed as cohomology of the complex $(\Omega^*G)^G$ of right invariant forms. The elements of the Lie algebra are right invariant vector fields, so one can dualize and get a complex computing de Rham cohomology of G in terms of the Lie algebra (Cartan's theorem). This complex is now known as the Koszul complex. Let us write it down a little more generally, namely with coefficients in a $\mathfrak{g}$ module M. (Think of a connection in a bundle.) So we must dualize the formulas for the de Rham complex (with values in a bundle with integrable connection?)

1.1 Koszul complex. We tensor the exterior algebra $\bigwedge \mathfrak{g}$ over the ground field k (usually $\mathbb{R}$ or $\mathbb{C}$) with M and define a differential ∂ by

$$\begin{aligned}
&\partial(a \otimes (g_1 \wedge \cdots \wedge g_q)) = \\
&\quad \textstyle\sum_{1\le s<t\le q} (-1)^{s+t-1} a \otimes ([g_s, g_t] \wedge g_1 \wedge \cdots \widehat{g_s} \cdots \widehat{g_t} \cdots \wedge g_q) + \\
&\quad \textstyle\sum_{1\le s\le q} (-1)^s g_s a \otimes (g_1 \wedge \cdots \widehat{g_s} \cdots \wedge g_q).
\end{aligned}$$

The homology groups of this complex are denoted $H_i(\mathfrak{g}, M)$, or simply $H_i(\mathfrak{g})$ if M is the module k with trivial action. If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of $\mathfrak{g}$ modules one gets a long exact sequence

$$\ldots H_i(\mathfrak{g}, M') \to H_i(\mathfrak{g}, M) \to H_i(\mathfrak{g}, M'') \to H_{i-1}(\mathfrak{g}, M') \ldots$$

and if M is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ one sees, by constructing an algebraic homotopy, that $H_i(\mathfrak{g}, \mathcal{U}(\mathfrak{g})) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$
This is used to show that

$$H_i(\mathfrak{g}, M) = \mathrm{Tor}_i^{\mathcal{U}(\mathfrak{g})}(k, M).$$

One may easily extend the definition of $H_i(\mathfrak{g}, M)$ to the case that M is not a single module, but a complex $M.$ of $\mathfrak{g}$ modules. For this one passes to the "total complex" of $M. \otimes \bigwedge^{\cdot} \mathfrak{g}$, by taking together the $M_i \otimes \bigwedge^j \mathfrak{g}$ with the same value of $i + j$. Then if $M. \to N.$ is a quasi-isomorphism, i.e. if it induces isomorphisms $H_i(M.) \to H_i(N.)$, one gets isomorphisms $H_i(\mathfrak{g}, M.) \to H_i(\mathfrak{g}, N.)$. In other words, the usual machinery of homological algebra (projective resolutions, spectral sequences, ...) applies.

As the cohomology of a compact Lie group was the first example of a Hopf algebra, it is not surprising that $H.(\mathfrak{g})$ is a Hopf algebra too. If $\mathfrak{g}$ is finite dimensional over $\mathbb{R}$, the structure theory of Hopf algebras thus predicts that $H.(\mathfrak{g})$ is an exterior algebra on odd dimensional generators, known as *primitives.* See [1].

It is similarly not surprising that $\mathfrak{g}$ acts trivially on $H^{\cdot}(\mathfrak{g})$.

There is a dual theory of Lie algebra *cohomology*, computing the groups $\mathrm{Ext}^i_{\mathcal{U}(\mathfrak{g})}(k, M)$ with a similar "Koszul complex". For finite dimensional M and $\mathfrak{g}$ one simply has $H^i(\mathfrak{g}, M^*) = H_i(\mathfrak{g}, M)^*$ where $*$ refers to the linear dual.

2 Examples

2.1 Semi-simple case. Let $\mathfrak{g}$ be a semi-simple Lie algebra. Note that the computation of Lie algebra homology commutes with extension of the base field, so that we may freely pass from $\mathbb{R}$ to $\mathbb{C}$ or to a convenient real form of $\mathfrak{g}$. By Cartan's theorem the computation of $H.(\mathfrak{g})$ amounts to the computation of the Betti or de Rham cohomology of the corresponding compact Lie group G. One may also try to compute directly by way of the Koszul complex. Borel did the computation in the topological setting and Koszul did it with the Koszul complex. Either way is non-trivial. For the case of the general linear groups things become a little more manageable, see Fuks. We postpone giving the answer, as it is best discussed in the context of spectral sequences.

Staying with our semi-simple Lie algebra, we can point out more structure. First of all $H_0(\mathfrak{g}, M)$, which in general is the module of co-invariants $M_{\mathfrak{g}}$, is now also the module of invariants $M^{\mathfrak{g}}$, provided M is finite dimensional. (Infinite dimensional counterexample: The universal enveloping algebra itself, which has coinvariants k via the augmentation map, has no invariants by Poincaré–Birkhoff–Witt. Note that the augmentation map also gives an example of a $\mathfrak{g}$ module map which does not split.)

More generally, one has

$$H_i(\mathfrak{g}, M) = H_i(\mathfrak{g}, M^{\mathfrak{g}})$$

for finite dimensional M, by the following reasoning. The center $Z(\mathfrak{g})$ acts on $\mathfrak{g}$ modules by module endomorphisms, so if M is an irreducible left $\mathfrak{g}$ module and N is an irreducible right $\mathfrak{g}$ module, we get two actions of $Z(\mathfrak{g})$ on $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{g})}(N, M)$, one according to the central character of M, the other according to the one of N. But elements of the center pass through the tensor product over $\mathcal{U}(\mathfrak{g})$, so these two actions must agree. It follows that all $\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{g})}(N, M)$ vanish unless the characters agree.

Thus we see that to understand $H_i(\mathfrak{g}, M)$ for finite dimensional M, the case that matters is $H_i(\mathfrak{g})$ itself. We can say that $H_1(\mathfrak{g})$ vanishes because $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and that $H_2(\mathfrak{g})$ vanishes because it classifies central extensions of $\mathfrak{g}$. (It is well known that such extensions split for semi-simple $\mathfrak{g}$.) But $H_3(\mathfrak{g})$ does not vanish, for the simple reason that one easily writes down a cocycle which is not a boundary, in terms of the Killing form and the Lie bracket. This will be an exercise as soon as one has made the following observations (cf. Bourbaki, Kostant). Let us work over $\mathbb{R}$. The Killing form puts a non-degenerate inner product on the $\bigwedge^i \mathfrak{g}$, so we can imitate Hodge–de Rham theory by defining an adjoint d of ∂, then a Laplace operator $\Delta = d\partial + \partial d$, then harmonic forms as those in $\ker \Delta$. It is not difficult to see that a form is harmonic if and only if it is invariant under the adjoint action. And, as is to be expected, on the subalgebra of harmonic forms the differential ∂ is trivial, while the inclusion $(\bigwedge \mathfrak{g})^{\mathfrak{g}} \to \bigwedge \mathfrak{g}$ is a quasi-isomorphism. So

$$H_i(\mathfrak{g}) = (\bigwedge^i \mathfrak{g})^{\mathfrak{g}},$$

and the problem is to compute those invariants in $\bigwedge^i \mathfrak{g}$.

2.2 Unipotent case. Now let us work over $\mathbb{C}$, and let $\mathfrak{u}$ be the Lie algebra of the unipotent radical U of a a Borel subgroup $B = TU$, of which I think as an algebraic group, as is my want. There is a pairing between the universal enveloping algebra $\mathcal{U}(\mathfrak{u})$ and the ring $k[U]$ of polynomial functions on U, assigning to the pair (D, f) the value of Df at the identity e. This makes the standard injective $k[U]$ in the category of (algebraic) representations of U into the T finite dual of the standard projective $\mathcal{U}(\mathfrak{u})$ in the category of $\mathfrak{u}$ modules $= \mathcal{U}(\mathfrak{u})$ modules. (The T finite dual is the largest subspace of the dual on which T acts as an algebraic group. It is just the sum of the duals of the weight spaces.) We are now interested in the computation of $H_{.}(\mathfrak{u}, M)$ when M is a (finite dimensional) simple G module with highest weight λ. Even though this confuses me, we order the weights so that the roots of B are *negative*. The result is given by Kostant's generalized Borel–Weil–Bott Theorem. Kostant himself computed with the Koszul complex. For a good exposition of this approach see Vogan's book. (One must pick out the components with trivial central character ...) Instead of staying with the Koszul complex we prefer to take the T finite dual which reduces the problem to that of computing $H^{\cdot}(U, M^*)$. That group can be understood rather well with the help of induction of representations from one algebraic group to another, as treated in Jantzen's book [5]. But again spectral sequences are used, so we postpone this too.

2.3 Lie super algebras. There is an analogue of the Koszul complex for Lie super-algebras. Recall, cf. Fuks, that a Lie superalgebra is a $\mathbb{Z}$ mod $\nvDash$ graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is called the even part, $\mathfrak{g}_1$ the odd part, together with a commutation operation [,] which satisfies

$$[g_1, g_2] = -(-1)^{p_1 p_2}[g_2, g_1],$$

$$(-1)^{p_1 p_3}[[g_1, g_2], g_3] + (-1)^{p_2 p_1}[[g_2, g_3], g_1] + (-1)^{p_3 p_2}[[g_3, g_1], g_2] = 0$$

for $g_i \in \mathfrak{g}_{p_i}$. The Koszul complex is now based not on $\bigwedge \mathfrak{g}$, but on its superanalogue $\bigwedge \mathfrak{g}_0 \otimes S\mathfrak{g}_1$, and the differential has a formula which is quite similar to the one for ordinary Lie algebras. It is too long and boring to reproduce here. See Fuks.

3 Spectral sequences

3.1 Homology of a filtered complex. Let

$$(C_{.}, d) = 0 \stackrel{d}{\leftarrow} C_0 \stackrel{d}{\leftarrow} C_1 \stackrel{d}{\leftarrow} \cdots$$

be a complex, of $\mathfrak{g}$ modules say, and let it be filtered by a decreasing sequence of subcomplexes which we call $C_{.}^{\geq p}$. Put $C_{.}^{\geq \infty} = \bigcap_p C_{.}^{\geq p}$ and $C_{.}^{\geq -\infty} =$

$\bigcup_p C_\cdot^{\geq p}$ and assume $C_\cdot^{\geq \infty}$ is the zero complex while $C_\cdot^{\geq -\infty} = C_\cdot$. In fact we wish to assume the following condition of *finite convergence*: For fixed degree n the filtration on C_n is finite, i.e. there is t so that $C_n^{\geq -t} = C_n^{\geq -\infty} = C_n$, $C_n^{\geq t} = C_n^{\geq \infty} = 0$. If $i \leq j$, we write $C_\cdot^{i/j}$ for the quotient complex $C_\cdot^{\geq i}/C_\cdot^{\geq j}$ and we filter the homology groups $H^p(C_\cdot^{i/j})$ by putting $H^p(C_\cdot^{i/j})^{\geq q} =$ image of $H^p(C_\cdot^{q/j})$ in $H^p(C_\cdot^{i/j})$.

Now one has a spectral sequence

$$E^2_{pq} = H_{p+q}(C_\cdot^{q/q+1}) \Rightarrow H_{p+q}(C_\cdot)$$

which tries to give an organized link between the homology of the successive filter-quotients $C_\cdot^{q/q+1}$ and the successive filter-quotients $H_{p+q}(C_\cdot)^{q/q+1}$ of the homology $H_{p+q}(C_\cdot) = H_{p+q}(C_\cdot^{-\infty/\infty})$. There is a plethora of long exact sequences trivially associated with this situation, and the spectral sequence makes an intelligent choice of notation in all this mess.

More specifically, if one puts $E^r_{pq} = (H_{p+q}(C_\cdot^{q-r+2/q+r-1}))^{q/q+1}$, then there is a natural differential d^r of total degree -1 on the bigraded group E^r, $r \geq 2$, so that the "term" E^{r+1} is just the homology $H(E^r, d^r)$. This differential is induced by the original differential of $C_\cdot$ and one has $d^r : E^r_{pq} \to E^r_{p-r,q+r-1}$.

For fixed p, q the E^r_{pq} may be identified with E^∞_{pq} for r sufficiently large. And $E^\infty_{pq} = H_{p+q}(C_\cdot)^{q/q+1}$ clearly gives information about the *abutment* $H_{p+q}(C_\cdot) = H(C_\cdot^{-\infty/\infty})$.

What is called a *spectral sequence* is this collection of bigraded groups and differentials. (Plus some isomorphisms like the one between E^{r+1} and $H_\cdot(E^r, d^r)$.)

One can do something entirely similar for cohomology instead of homology: The morphisms are reversed and E^r_{pq} becomes E_r^{pq}.

3.2 The Hochschild–Serre spectral sequence. Let us just describe it for the case of a Lie algebra $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{u}$ where $\mathfrak{u}$ is an ideal in $\mathfrak{l}$, and $\mathfrak{g}$ is a complementary subalgebra. The Koszul complex for $\mathfrak{l}$ has an obvious filtration by subcomplexes: Put $(\bigwedge^n \mathfrak{l})^{\geq -r} = \sum_{j=1}^r \bigwedge^j \mathfrak{g} \otimes \bigwedge^{n-j} \mathfrak{u}$. We thus get a spectral sequence with $\tilde{E}^2_{pq} = H_{p+q}(\bigwedge^{-q} \mathfrak{g} \otimes \bigwedge^\cdot \mathfrak{u})$, where $\bigwedge^{-q} \mathfrak{g} \otimes \bigwedge^i \mathfrak{u}$ sits in degree $i - q$. As that leads to a rather stupid and unconventional indexing, we put $E^r_{pq} = \tilde{E}^{r+1}_{2p+q,-p}$, and get the better looking

$$E^2_{pq} = H_p(\mathfrak{g}, H_q(\mathfrak{u})) \Rightarrow H_{p+q}(\mathfrak{l}).$$

Now suppose $\mathfrak{l}$ is finite dimensional and $\mathfrak{g}$ is semi-simple. Then $E^2_{pq} = H_p(\mathfrak{g}, H_q(\mathfrak{u})^{\mathfrak{g}})$, and because taking $\mathfrak{g}$ fixed points is exact on finite dimensional modules, $H_q(\mathfrak{u})^{\mathfrak{g}} = H_q((\bigwedge \mathfrak{u})^{\mathfrak{g}})$. This makes us suspect that we should consider the subcomplex $(\bigwedge \mathfrak{g}) \otimes (\bigwedge \mathfrak{u})^{\mathfrak{g}}$ of the Koszul complex of $\mathfrak{l}$. Incidentally, this subcomplex is in fact a subalgebra, and what we are doing here is exploiting the *multiplicative structure* on the terms of the Hochschild–Serre spectral sequence. The subcomplex inherits a filtration from the one on $\bigwedge \mathfrak{l}$ and one thus gets a *morphism of spectral sequences* which visibly induces an isomorphism at the E^2 level. But then it must also induce an isomorphism of abutments. The upshot is the theorem of Hochschild–Serre

$$H.(\mathfrak{l}) = H.(\mathfrak{g}) \otimes (H.(\mathfrak{u}))^{\mathfrak{g}},$$

as the right hand side is easily seen to be the homology of our subcomplex.

3.3 A Grothendieck spectral sequence. The Hochschild–Serre spectral sequence above can also be understood as a Grothendieck spectral sequence for composite functors. Namely $H_i(\mathfrak{l}, M)$ is the i-th derived functor of $M \mapsto M_{\mathfrak{l}}$, while $M \mapsto M_{\mathfrak{l}}$ is the composite of the functors $N \mapsto N_{\mathfrak{u}}$ and $L \mapsto L_{\mathfrak{g}}$. We may compute $H_i(\mathfrak{u})$ as the i-th homology of the complex $C. = (P.)_{\mathfrak{u}}$, where $P.$ is a projective resolution of k as $\mathcal{U}(\mathfrak{l})$ module, and thus also a projective resolution of k as $\mathcal{U}(\mathfrak{u})$ module. Not that the C_n are projective $\mathcal{U}(\mathfrak{g})$ modules, so that one does not need to resolve $C.$ to compute $H_i(\mathfrak{g}, C.)$. One has $H_i(\mathfrak{g}, C.) = H_i((C.)_{\mathfrak{g}}) = H_i((P.)_{\mathfrak{l}}) = H_i(\mathfrak{l})$. This will explain the abutment of our spectral sequence. To get the levels, we put a *homological filtration* (cf. "truncation" or "t-structure") on $C.$:

$$C_n^{\geq q} = \begin{cases} 0 & \text{if } n < q \\ \ker d & \text{if } n = q \\ C_n & \text{if } n > q \end{cases}$$

Then $C.^{q/q+1}$ is quasi-isomorphic to a complex that is concentrated in degree q, which explains the E^2 term in

$$E^2_{pq} = H_p(\mathfrak{g}, H_q(\mathfrak{u})) \Rightarrow H_{p+q}(\mathfrak{l}).$$

For the general level we simply imitate the earlier formulas: $E^r_{pq} = (H_{p+q}(\mathfrak{g}, C.^{q-r+2/q+r-1}))^{q/q+1}$. As this example shows, those earlier formulas are just one case of a more general construction, which is incidentally best understood in terms of the exact couples of Massey.

4 Computations

4.1 Semi-simple case. We return to the problem of computing $H_\cdot(\mathfrak{g})$ for semi-simple $\mathfrak{g}$. We recommend [4], [1] and [2] for further information. There are two related approaches, both leading to the answer that $H_\cdot(\mathfrak{g})$ is a tensor product of exterior algebras on primitive generators x_i, with x_i living in homological degree $2m_i + 1$ where m_i is the i-th *exponent* of the Weyl group. Indeed x_i arises as the transgression (=image under a differential which starts at $E^r_{0,r}$ and ends at $E^r_{0,r-1}$) of the generator in degree $m_i + 1$ (see [3]) of the invariant ring $(S^\cdot(\mathfrak{t}))^W$. The homological degree of the $\mathfrak{t}$ in $(S^\cdot(\mathfrak{t}))$ happens to be 2, so the homological degree of the generator is $2m_i + 2$, which explains why x_i has homological degree $2m_i + 1$.

In the case of Koszul, the spectral sequence in which all this happens is the one associated to a filtration of the "Weil algebra", which is a differential graded algebra, quasi-isomorphic to k, with underlying algebra $(\bigwedge^\cdot \mathfrak{g}) \otimes (S^\cdot \mathfrak{g})$, with the first $\mathfrak{g}$ in degree 1, the second in degree 2. One encounters $(S^\cdot(\mathfrak{t}))^W$ in the guise of $(S^\cdot(\mathfrak{g}^*))^{\mathfrak{g}}$. All this is very algebraic.

In Borel's computation the spectral sequence is the Leray spectral sequence

$$E_2^{pq} = H^p(B_G, H^q(G)) \Rightarrow H^{p+q}(E_G) = H^{p+q}(\text{point})$$

for the cohomology (we have dualized) of the principal fibration $E_G \to B_G = E_G/G$ where E_G is the universal bundle of the compact Lie group G, and B_G is the classifying space of G. He needs to compare it with the analogous spectral sequence for a maximal torus T of G, and this involves a study of G/T, the fiber of $B_G \to B_T$. (One uses that G/T has cohomology only in even degrees, which follows for instance from the fact that it can be viewed as a complex flag variety. This "odd vanishing" enables one to determine the character of $H^\cdot(G/T)$, as a W module, by means of the Lefschetz fixed point formula.) The connection with exponents then comes through the fact that $H^\cdot(B_T)$ is a polynomial algebra on which the Weyl group acts, with $H^\cdot(B_T)^W$ mapping isomorphically to $H^\cdot(B_G)$. We can not give more details here.

4.2 Kostant's generalized Borel–Weil–Bott theorem. Now let G, B, T, U be algebraic groups, and let a notation like $H^\cdot(G, M)$ now refer to $\mathrm{Ext}^\cdot_G(k, M)$, the ext groups in the category of (algebraic) representations of G. So $H^\cdot(G)$ no longer refers to de Rham cohomology like it did up till now.

Recall we are interested in $H^{\cdot}(U, M)$ where M is an irreducible, hence finite dimensional, G-module.

A key tool here is the notion of *induction* which is appropriate in this theory. Unlike induction in the representation theory of discrete groups, induction is now the *right* (not left) adjoint of restriction. Thus Frobenius reciprocity now reads:

$$\operatorname{Hom}_H(\operatorname{Res}_H^K M, N) = \operatorname{Hom}_K(M, \operatorname{Ind}_H^K N),$$

or more generally

$$\operatorname{Hom}_H(\operatorname{Res}_H^K M, N \otimes \operatorname{Res}_H^K L) = \operatorname{Hom}_K(M, (\operatorname{Ind}_H^K N) \otimes L),$$

where Hom_K refers to the group of homomorphisms of (algebraic) K modules. If N is finite dimensional, $\operatorname{Ind}_H^K N$ is constructed as the module of global sections over the coset space K/H of the unique K equivariant vector bundle $\mathcal{L}(N)$ whose fiber over the point H/H is N. In particular, if P is a minimal parabolic subgroup containing B, then P/B is a projective line, so $\operatorname{Ind}_B^P k = k$. (Similarly $\operatorname{Ind}_P^G k = k$.) It follows that $\operatorname{Ind}_B^P \operatorname{Res}_B^P = \mathrm{id}$. This makes that $\operatorname{Ind}_B^G = \operatorname{Ind}_P^G \operatorname{Ind}_B^P = \operatorname{Ind}_P^G \operatorname{Ind}_B^P \operatorname{Res}_B^P \operatorname{Ind}_B^P = \operatorname{Ind}_B^G \operatorname{Res}_B^P \operatorname{Ind}_B^P$ or, if we drop Res_B^P from notations,

$$\operatorname{Ind}_B^G = \operatorname{Ind}_B^G \operatorname{Ind}_B^P .$$

This leads to a Grothendieck spectral sequence to study the derived functors $R^i \operatorname{Ind}_B^G$ in terms of themselves and the much simpler $R^i \operatorname{Ind}_B^P$. (They are much simpler because we are talking cohomology on a projective line $\mathbb{P}^{\Bbbk}$.) From Frobenius reciprocity we know that if λ is a character of B (or T), then $\operatorname{Ind}_B^G \lambda \neq 0$ if and only if there is a G module with highest weight λ. (In fact $\operatorname{Ind}_B^G \lambda$ is then the irreducible G module with high weight λ.) On the other hand we know that $R^i \operatorname{Ind}_B^G$ is zero for $i > \dim(G/B)$. Combining these observations with the study of the $R^i \operatorname{Ind}_B^P$ for all minimal parabolics leads to an easy proof (due to Demazure) of:

Borel–Weil–Bott Theorem.
Let λ be dominant.

$$R^i \operatorname{Ind}_B^G(w \cdot \lambda) = \begin{cases} \operatorname{Ind}_B^G(\lambda) & \text{if } i = \ell(w) \\ 0 & \text{otherwise} \end{cases}$$

Here $w \cdot \lambda = w(\lambda + \rho) - \rho$, as usual.

The appearance both of ρ and of the reflections is explained here by Serre duality on the projective line: If α is the simple root which is a root of P, with corresponding simple reflection s, then it turns out that the "canonical line bundle" of 1-forms is just $\mathcal{L}(-\alpha)$ on P/B and Serre duality thus becomes

$$R^i \operatorname{Ind}_B^P \mu = (R^{1-i} \operatorname{Ind}_B^P(-\alpha - \mu))^*.$$

If $R^1 \operatorname{Ind}_B^P \mu \neq 0$ this means that $-\alpha-\mu$ is dominant with respect to the Levi factor, say SL_2, of P, so $(-\alpha - \mu, \alpha) > 0$ and $(R^1 \operatorname{Ind}_B^P \mu)^*$ is the irreducible SL_2 module with highest weight $-\alpha - \mu$. Thus $R^1 \operatorname{Ind}_B^P \mu$ is the irreducible SL_2 module $\operatorname{Ind}_B^P(s(-\alpha - \mu))$ with highest weight $s(-\alpha - \mu)$. That is,

$$R^1 \operatorname{Ind}_B^P \mu = \operatorname{Ind}_B^P(s.\mu).$$

We see here how a reflection in the "dot action" corresponds with a degree shift in sheaf cohomology. For further details on this proof of the Borel–Weil–Bott Theorem we refer to Jantzen's book [5].

Returning to our problem of computing $H^{\cdot}(U, M)$ we observe that by Frobenius reciprocity $N^U = (\operatorname{Ind}_U^G N)^G$ so that we have a Grothendieck spectral sequence

$$H^p(G, R^q \operatorname{Ind}_U^G M)) \Rightarrow H^{p+q}(U, M).$$

The algebraic representation theory of G enjoys complete reducibility, so the spectral sequence simplifies ("degenerates") to

$$(R^q \operatorname{Ind}_U^G M)^G = H^q(U, M).$$

Further $R^q \operatorname{Ind}_U^G M = R^q \operatorname{Ind}_B^G \operatorname{Ind}_U^B M$ for similar reasons: Ind_U^B is an exact functor because $B/U = T$ is an affine scheme, or more concretely, because $\operatorname{Ind}_U^B N = \sum_\lambda \lambda \otimes N$ as vector spaces, and even as B modules if N is the restriction of a B module. (The summation is over all weights.) Taking all this together we come to the following computation: $H^i(U, M)$ is the sum over all dominant λ and all $w \in W$ with $\ell(w) = i$ of the $(\operatorname{Ind}_B^G(\lambda) \otimes M)^G = \operatorname{Hom}_G(M^*, \operatorname{Ind}_B^G(\lambda)) = \operatorname{Hom}_B(M^*, \lambda)$. Clearly the only λ which can contribute is the highest weight of M^*. Thus

Kostant's Theorem.

Let M be an irreducible G module. Then

$$\dim H^i(U, M) = \#\{w \in W \mid \ell(w) = i\}$$

and the total dimension of $H^{\cdot}(U, M)$ is the order of the Weyl group.

Remark. Kostant not only treats the Lie algebra of the unipotent radical of B, but of any parabolic subgroup. We could have done the same, but for general Levi factors L the decomposition [5, II 4.20] of the regular representation in the algebra of "representative functions" $k[L]$ is not as trivial as for a torus. ("Peter-Weyl theory".)

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