BIFUNCTOR COHOMOLOGY AND COHOMOLOGICAL FINITE GENERATION FOR REDUCTIVE GROUPS

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Abstract

Let G be a reductive linear algebraic group over a field k. Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. Invariant theory states that the ring of invariants $A^G = H^0(G, A)$ is finitely generated. We show that in fact the full cohomology ring $H^*(G, A)$ is finitely generated. The proof is based on the strict polynomial bifunctor cohomology classes constructed in [22]. We also continue the study of bifunctor cohomology of $\Gamma^*(\mathfrak{gl}^{(1)})$.

1. Introduction

Consider a linear algebraic group G, or a linear algebraic group scheme G, defined over a field k. So G is an affine group scheme whose coordinate algebra $k[G]$ is finitely generated as a k-algebra. We say that G has the cohomological finite generation (CFG) property if the following holds. Let A be a finitely generated commutative k-algebra on which G acts rationally by k-algebra automorphisms. (So G acts from the right on $\operatorname{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a k-algebra.

Here, as in [12, Part I, Section 4], we use the cohomology introduced by Hochschild, also known as *rational cohomology*.

Our main result confirms a conjecture of the second author, as follows.

THEOREM 1.1
Any reductive linear algebraic group over k has the CFG property.

The proof is based on the first author's "lifted" universal classes (see [22]), which were constructed for this purpose. Originally [22] was the end of the proof, but for the purpose of exposition we have changed the order.

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If the field k has characteristic zero, then the theorem just reiterates a standard fact in invariant theory. Indeed, the reductive group is then linearly reductive, and rational cohomology vanishes in higher degrees for any linearly reductive group.

So we further assume that k has positive characteristic p. In this introduction, we also take k algebraically closed. (One easily reduces to this case; cf. [24, Lemma 2.3], [12, Part I, Section 4.13], [25].) We say that G acts on the algebra A if G acts rationally by k-algebra automorphisms.

Let us say that G has the finite generation (FG) property, or is a positive solution to Hilbert's 14th problem, if the following holds. If G acts on a finitely generated commutative k-algebra A, then the ring of invariants $A^G = H^0(G, A)$ is finitely generated as a k-algebra. Observe that, unlike Hilbert, we do *not* require that A be a domain.

It is obvious that CFG implies FG. We see that our main result can also be formulated as follows.

THEOREM 1.2
A linear algebraic group scheme G over k has the CFG property if and only if it has the FG property.

Let us give some examples. The first example is a finite group G, viewed as a discrete algebraic group over k. It is well known to have the FG property (see [24, Lemma 2.4]), and the proof goes back to Noether's 1926 proof [17]. Thus, we recover the FG theorem of Evens, at least over our field k.

THEOREM 1.3 (Evens [6, Theorem 8.1])
A finite group has the CFG property over k.

As our proof of Theorem 1.2 does not rely on Theorem 1.3, we get a new proof of the latter, albeit much longer than the original proof. Note that the setting of Evens is more general: instead of a field, he allows an arbitrary Noetherian base. This suggests a direction for further work.

If G is a linear algebraic group over k, we write G_r for its rth Frobenius kernel, the scheme theoretic kernel of the rth iterate $F^r : G \to G^{(r)}$ of the Frobenius homomorphism (see [12, Part I, Chapter 9]). It is easy to see that G_r has the FG property. More generally, it is easy to see (cf. [24, Lemma 2.4]) that any finite group scheme over k has the FG property. (A group scheme is finite if its coordinate ring is a finite-dimensional vector space.) Indeed, one has [24, Theorem 3.5].

THEOREM 1.4 (Friedlander and Suslin [9])
A finite group scheme has the CFG property.

But we do not get a new proof of this theorem, as our proof of the main result relies heavily on the specific information in [9, Section 1]. Recall that the theorem of Friedlander and Suslin was motivated by a desire to get a theory of support varieties for infinitesimal group schemes. Our problem has the same origin. Then it started to get a life of its own and became a conjecture.

It is a theorem of Nagata [16, Main Theorem], [19, Chapter 2] that geometrically reductive groups (or group schemes; see [3]) have the FG property. (Springer [19] deletes "geometrically" in the terminology.) Conversely, it is elementary (see [24, Theorem 2.2]) that the FG property implies geometric reductivity. (Here it is essential that, in the FG propery, one allows any finitely generated commutative k-algebra on which G acts.) So our main result states that the CFG property is equivalent to geometric reductivity.

Now, Haboush has shown in [12, Part II, Section 10.7] that reductive groups are geometrically reductive, and Popov [18] has shown that a linear algebraic group with the FG property is reductive. (Popov allows only reduced algebras, so his result is even stronger.) Waterhouse [25] has completed the classification by showing that a linear algebraic group scheme G (he calls it an "algebraic affine group scheme") is geometrically reductive exactly when the connected component G^o_{red} of its reduced subgroup G_{red} is reductive. So this is also a characterization of the G with the CFG property.

Let us now give a consequence of CFG. We say that G acts on an A-module M when it acts rationally on M such that the structure map $A \otimes M \to M$ is a G-module map.

THEOREM 1.5
Let G have the CFG property. Let G act on the finitely generated commutative k-algebra A and on the Noetherian A-module M. Then $H^*(G, M)$ is a Noetherian $H^*(G, A)$-module. In particular, if G is reductive and A has a good filtration, then $H^*(G, M)$ is a Noetherian A^G-module, $H^i(G, M)$ vanishes for large i, and M has finite good filtration dimension.

Proof
See [24, Lemma 3.3, proof of Corollary 4.7]. One puts an algebra structure on $A \oplus M$ and uses the fact that $A \otimes k[G/U]$ also has a good filtration. □

As a special case, we mention the following.

THEOREM 1.6
Let $G = \mathrm{GL}_n$, $n \geq 1$. *Let G act on the finitely generated commutative k-algebra A and on the Noetherian A-module M. If A has a good filtration, then $H^*(G, M)$*

is a Noetherian A^G-module, $H^i(G, M)$ vanishes for large i, and M has finite good filtration dimension.

This theorem is proved directly in [20, Theorem 1.1], with functorial resolution of the ideal of the diagonal in a product of Grassmannians; we use it in our proof of the main theorems.

Now let us start discussing the proof of the main result. First of all, one has the following variation on the ancient transfer principle (see [11, Chapter 2]).

LEMMA 1.7 ([24, Lemma 3.7])
Let G be a linear algebraic group over k with the CFG property. Then any geometrically reductive subgroup scheme H of G also has the CFG property.

As every geometrically reductive linear algebraic group scheme is a subgroup scheme of GL_n for n sufficiently large, we only have to look at the GL_n to prove the main theorems, Theorem 1.1 and Theorem 1.2. Therefore, we further assume that $G = \mathrm{GL}_n$ with $n > 1$ (or with $n \geq p$, if you wish.) (In [23], we used SL_n instead of $G = \mathrm{GL}_n$, but we also explained that it hardly makes any difference.)

We have G act on A, and we wish to show that $H^*(G, A)$ is finitely generated. If A has a good filtration (see [12]), then there is no higher cohomology and invariant theory (Haboush) does the job. A general A has been related by Grosshans [10] to one with a good filtration. He defines a filtration $A_{\leq 0} \subseteq A_{\leq 1} \cdots$ on A and embeds the associated graded $\mathrm{gr}\, A$ into an algebra with a good filtration $\mathrm{hull}_\nabla \mathrm{gr}\, A$. He shows that $\mathrm{gr}\, A$ and $\mathrm{hull}_\nabla \mathrm{gr}\, A$ are also finitely generated and that there is a flat family parametrized by the affine line with special fiber $\mathrm{gr}\, A$ and general fiber A. We write $\mathscr{A}$ for the coordinate ring of the family. It is a graded algebra, and one has natural homomorphisms $\mathscr{A} \to \mathrm{gr}\, A$, $\mathscr{A} \to A$. Mathieu has shown in [15] (cf. [23, Lemma 2.3]) that there is an $r > 0$ so that $x^{p^r} \in \mathrm{gr}\, A$ for every $x \in \mathrm{hull}_\nabla \mathrm{gr}\, A$. We have no bound on r, which is the main reason that our results are only qualitative. One sees that $\mathrm{gr}\, A$ is a Noetherian module over the rth Frobenius twist $(\mathrm{hull}_\nabla \mathrm{gr}\, A)^{(r)}$ of $\mathrm{hull}_\nabla \mathrm{gr}\, A$. So we do not quite have the situation of Theorem 1.6, but it is close. We have to untwist. Untwisting involves $G^{(r)} = G/G_r$, and we end up looking at the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i\big(G/G_r, H^j(G_r, \mathrm{gr}\, A)\big) \Rightarrow H^{i+j}(G, \mathrm{gr}\, A).$$

One may write $H^i(G/G_r, H^j(G_r, \mathrm{gr}\, A))$ also as $H^i(G, H^j(G_r, \mathrm{gr}\, A)^{(-r)})$. By Friedlander and Suslin, $H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ is a Noetherian module over the graded algebra $\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathrm{hull}_\nabla \mathrm{gr}\, A$. Here the # sign refers to taking a dual, S^* refers to a symmetric algebra over k, and the $(2p^{i-1})$ indicates in what degree one puts a copy of the dual of the adjoint representation $\mathfrak{gl}_n$. By the fundamental work of Akin, Buchsbaum, and Weyman [1], which is also of essential importance in [20], one knows that $\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathrm{hull}_\nabla \operatorname{gr} A$ has a good filtration. So $H^*(G_r, \operatorname{gr} A)^{(-r)}$ has finite good filtration dimension and page 2 of our Hochschild-Serre spectral sequence is Noetherian over its first column E_2^{0*}. By Friedlander and Suslin, $H^*(G_r, \operatorname{gr} A)^{(-r)}$ is a finitely generated algebra and, by invariant theory, E_2^{0*} is thus finitely generated, so E_2^{**} is finitely generated. The spectral sequence is one of graded commutative differential graded algebras in characteristic p, so the pth power of an even cochain in a page passes to the next page. It easily follows that all pages are finitely generated. As page 2 has only finitely many columns by Theorem 1.6 (see [20, Definition 2.3]), this explains why the abutment $H^*(G, \operatorname{gr} A)$ is finitely generated. We are getting closer to $H^*(G, A)$.

The filtration $A_{\le 0} \subseteq A_{\le 1} \cdots$ induces a filtration of the Hochschild complex [12, Part I, Section 4.14], whence comes a spectral sequence

$$E(A) : E_1^{ij} = H^{i+j}(G, \operatorname{gr}_{-i} A) \Rightarrow H^{i+j}(G, A).$$

It lives in the second quadrant, but as E_1^{**} is a finitely generated k-algebra, this causes no difficulty with convergence: given m, there are only finitely many nonzero $E_1^{m-i,i}$ (cf. [23, Section 4.11]; note that in [23], the E_1 page is mistaken for an E_2 page). All pages are again finitely generated, so we would like the spectral sequence to stop, meaning that $E_s^{**} = E_\infty^{**}$ for some finite s. There is a standard method to achieve this (see [6], [9]). One must find a "ring of operators" acting on the spectral sequence and show that some page is a Noetherian module for the chosen ring of operators. As the ring of operators, we take $H^*(G, \mathscr{A})$. Indeed, $E(A)$ is acted on by the trivial spectral sequence $E(\mathscr{A})$, whose pages equal $H^*(G, \mathscr{A})$ (see [23, Section 4.11]). And $H^*(G, \mathscr{A})$ also acts on our Hochschild-Serre spectral sequence through its abutment. If we can show that one of the pages of the Hochschild-Serre spectral sequence is a Noetherian module over $H^*(G, \mathscr{A})$, then that does the trick, as then the abutment $H^*(G, \operatorname{gr} A)$ is Noetherian by [9, Lemma 1.6]. And this abutment is the first page of $E(A)$.

Now we are in a situation similar to the one encountered by Friedlander and Suslin. Their problem was "surprisingly elusive." To make their breakthrough, they had to invent strict polynomial functors. Studying the homological algebra of strict polynomial functors, they found universal cohomology classes $e_r \in H^{2p^{r-1}}(G, \mathfrak{gl}_n^{(r)})$ with nontrivial restriction to G_1. That was enough to get through. We faced a similar bottleneck. We know from invariant theory and from [9] that page 2 of our Hochschild-Serre spectral sequence is Noetherian over $H^0\big(G, (\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathscr{A})^{(r)}\big)$. But we want it to be Noetherian over $H^*(G, \mathscr{A})$. So if we could factor the homomorphism $H^0\big(G, (\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathscr{A})^{(r)}\big) \to E_2^{0*}$ through $H^*(G, \mathscr{A})$, then that would do it. The universal classes e_j provide such a factorization on some summands, but they do not seem to help on the rest. One would like to have universal classes in more degrees so that one can map every summand of the form $H^0\big(G, (\bigotimes_{i=1}^r S^{m_i}((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathscr{A})^{(r)}\big)$ into the appropriate $H^{2m}(G, \mathscr{A})$, or even into $H^{2m}(G, \mathscr{A}^{G_r})$. The dual of $S^{m_i}((\mathfrak{gl}_n)^\#)^{(r)}$ is $\Gamma^{m_i}(\mathfrak{gl}_n^{(r)})$. Thus, one seeks nontrivial classes in $H^{2mp^{i-1}}(G, \Gamma^m(\mathfrak{gl}_n^{(r)}))$ to take cup product with. It turns out that $r = i = 1$ is the crucial case and we seek nontrivial classes $c[m] \in H^{2m}(G, \Gamma^m(\mathfrak{gl}_n^{(1)}))$. The construction of such classes $c[m]$ has been a sticking point at least since 2001. In [23], they were constructed for GL_2, but one needs them for GL_n with n large. The strict polynomial functors of Friedlander and Suslin do not provide a natural home for this problem, but the strict polynomial bifunctors of Franjou and Friedlander [8] do.

When the first author [22] found a construction of nontrivial "lifted" classes $c[m]$, this finished a proof of the conjecture. We present two proofs. The first proof continues the investigation of bifunctor cohomology in [22] and establishes properties of the $c[m]$ analogous to those employed in [23]. Then the result follows as in the proof in [23] for GL_2. As a byproduct, one also obtains extra bifunctor cohomology classes and relations between them. The second proof needs no more properties of the classes $c[m]$ than those established in [22]. Indeed, [22] stops exactly where the two arguments start to diverge. The second proof does not quite factor the homomorphism $H^0\big(G, (\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n)^\#(2p^{i-1})) \otimes \mathscr{A})^{(r)}\big) \to E_2^{0*}$ through $H^*(G, \mathscr{A})$, but argues by induction on r, returning to [9, Section 1] with the new classes in hand. It is not hard to guess which author contributes which proof. The first author goes first.

Part I. The first proof

3. Main theorem and CFG

We work over a field k of positive characteristic p. We keep the notation of [22]. In particular, $\mathscr{P}_k(1,1)$ denotes the category of strict polynomial bifunctors of [8]. The main result of part I is Theorem 3.1, which states the existence of classes in the cohomology of the bifunctor $\Gamma^*(\mathrm{gl}^{(1)})$. By [22, Theorem 1.3], the cohomology of a bifunctor B is related to the cohomology of $\mathrm{GL}_{n,k}$ with coefficients in the rational representation $B(k^n, k^n)$ by a map $\phi_{B,n} : H^*_{\mathscr{P}}(B) \to H^*(\mathrm{GL}_{n,k}, B(k^n, k^n))$ (natural in B and compatible with cup products). So our main result yields classes in the cohomology of $\mathrm{GL}_{n,k}$, actually more classes (and more relations between them) than originally needed (see [23, Section 4.3]) for the proof of the CFG conjecture.

THEOREM 3.1
Let k be a field of characteristic $p > 0$. There are maps $\psi_\ell : \Gamma^\ell H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \to H^*_{\mathscr{P}}(\Gamma^\ell(\mathrm{gl}^{(1)}))$, $\ell \geq 1$, such that the following hold.

(1) *The map ψ_1 is the identity map.*

(2) *For all $\ell \geq 1$ and for all $n \geq p$, the composite*

$$\Gamma^\ell H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \xrightarrow{\psi_\ell} H^*_{\mathscr{P}}\big(\Gamma^\ell(\mathrm{gl}^{(1)})\big) \xrightarrow{\phi_{\Gamma^\ell(\mathrm{gl}^{(1)}),n}} H^*\big(\mathrm{GL}_{n,k}, \Gamma^\ell(\mathfrak{gl}_n^{(1)})\big)$$

is injective. In particular, for all $\ell \geq 1$, ψ_ℓ is injective.

(3) *For all positive integers ℓ, m, there are commutative diagrams*

$$\begin{array}{ccc} H^*_{\mathscr{P}}\big(\Gamma^{\ell+m}(\mathrm{gl}^{(1)})\big) & \xrightarrow{\Delta_{\ell,m*}} & H^*_{\mathscr{P}}\big(\Gamma^{\ell}(\mathrm{gl}^{(1)})\otimes\Gamma^{m}(\mathrm{gl}^{(1)})\big) \\ \uparrow{\scriptstyle \psi_{\ell+m}} & & \uparrow{\scriptstyle \psi_\ell\cup\psi_m} \\ \Gamma^{\ell+m}H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) & \xhookrightarrow{\Delta_{\ell,m}} & \Gamma^{\ell}H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})\otimes\Gamma^{m}H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \end{array}$$

and

$$\begin{array}{ccc} H^*_{\mathscr{P}}\big(\Gamma^{\ell}(\mathrm{gl}^{(1)})\otimes\Gamma^{m}(\mathrm{gl}^{(1)})\big) & \xrightarrow{m_{\ell,m*}} & H^*_{\mathscr{P}}\big(\Gamma^{\ell+m}(\mathrm{gl}^{(1)})\big) \\ \uparrow{\scriptstyle \psi_\ell\cup\psi_m} & & \uparrow{\scriptstyle \psi_{\ell+m}} \\ \Gamma^{\ell}H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})\otimes\Gamma^{m}H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) & \xrightarrow{m_{\ell,m}} & \Gamma^{\ell+m}H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \end{array}$$

where $m_{\ell,m}$ and $\Delta_{\ell,m}$ denote the maps induced by the multiplication $\Gamma^\ell\otimes\Gamma^m \to \Gamma^{\ell+m}$ and the diagonal $\Gamma^{\ell+m} \to \Gamma^\ell\otimes\Gamma^m$.

As a consequence, we obtain that [23, Theorem 4.4] is valid for any value of n.

COROLLARY 3.2

Let k be a field of positive characteristic. For all $n > 1$, there are classes $c[m] \in H^{2m}(\mathrm{GL}_{n,k}, \Gamma^m(\mathfrak{gl}_n^{(1)}))$ such that

(1) *$c[1]$ is the Witt vector class e_1;*

(2) $\Delta_{i,j*}(c[i+j]) = c[i]\cup c[j]$ *for* $i, j \geq 1$.

Proof

Arguing as in [22, Lemma 1.5], we notice that it suffices to prove the statement when $n \geq p$. By [22, Theorem 1.3], we have morphisms

$$\phi_{\Gamma^m(\mathrm{gl}^{(1)}),n} : H^*_{\mathscr{P}}\big(\Gamma^m(\mathrm{gl}^{(1)})\big) \to H^*\big(\mathrm{GL}_{n,k}, \Gamma^m(\mathfrak{gl}_n^{(1)})\big)$$

compatible with the cup products, and for $m = 1$, the map $\phi_{\Gamma^m(\mathrm{gl}^{(1)}),n}$ is an isomorphism. Let $b[1]$ be the preimage of the Witt vector class by $\phi_{\Gamma^1(\mathrm{gl}^{(1)}),n}$. We define $c[m] := (\phi_{\Gamma^m(\mathrm{gl}^{(1)}),n}\circ\psi_m)(b[1]^{\otimes m})$. Then $c[1]$ is the Witt vector class since ψ_1 is the identity, and by Theorem 3.1(3), the classes $c[i]$ satisfy condition (2). □

COROLLARY 3.3

The CFG conjecture (Theorem 1.1) holds.

Proof

Let G be a reductive linear algebraic group acting on a finitely generated commutative k-algebra A. We want to prove that $H^*(G, A)$ is finitely generated. To do this, it suffices to follow [23], and this is exactly what we do below. We keep the notation of the introduction.

By Lemma 1.7, the case $G = \mathrm{GL}_{n,k}$ suffices. As recalled in the introduction, there exists a positive integer r such that the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i\big(G/G_r, H^j(G_r, \operatorname{gr} A)\big) \Rightarrow H^{i+j}(G, \operatorname{gr} A)$$

stops for a finite good filtration dimension reason. Moreover, it is a sequence of finitely generated algebras, and its second page is Noetherian over its subalgebra E_2^{0*} (all this was first proved in [23, Proposition 3.8] under some restrictions on the characteristic which were removed in [20]).

The composite $\mathscr{A}^{G_r} \hookrightarrow \mathscr{A} \twoheadrightarrow \operatorname{gr} A$ makes $\operatorname{gr} A$ into a Noetherian module over $\mathscr{A}^{G_r}$. Hence, by [9, Theorem 1.5] (with "C" $= \mathscr{A}^{G_r}$) and by invariant theory (see [11, Theorem 16.9]), $E_2^{0*} = H^0(G/G_r, H^*(G_r, \operatorname{gr} A))$ (hence E_2^{**}) is Noetherian over $H^0\big(G/G_r, \bigotimes_{i=1}^r S^*((\mathfrak{gl}_n^{(r)})^\#(2p^{i-1})) \otimes \mathscr{A}^{G_r}\big)$.

Now we use the classes of Corollary 3.2 as in Section 4.5 and as in the proof of [23, Corollary 4.8]. In this way, we factor the morphism $H^0\big(G/G_r, \bigotimes_{i=1}^r S^*((\mathfrak{gl}_n^{(r)})^\#(2p^{i-1})) \otimes \mathscr{A}^{G_r}\big) \to E_2^{0*}$ through the map $H^{\mathrm{even}}(G, \mathscr{A}) \to H^0(G/G_r, H^{\mathrm{even}}(G_r, \mathscr{A})) = E_2^{0\,\mathrm{even}}$. (The latter map is induced by restricting the cohomology from G to G_r.) So E_2^{**} is Noetherian over $H^{\mathrm{even}}(G, \mathscr{A})$. By [9, Lemma 1.6] (with "A" $= H^{\mathrm{even}}(G, \mathscr{A})$ and "B" $= k$), we conclude that the map $H^{\mathrm{even}}(G, \mathscr{A}) \to H^*(G, \operatorname{gr} A)$ (induced by $\mathscr{A} \to \operatorname{gr} A$) makes $H^*(G, \operatorname{gr} A)$ into a Noetherian module over $H^{\mathrm{even}}(G, \mathscr{A})$.

The proof finishes as described in the introduction (or in [23, Section 4.11]): the second spectral sequence

$$E(A) : E_1^{ij} = H^{i+j}(G, \operatorname{gr}_{-i} A) \Rightarrow H^{i+j}(G, A)$$

is a sequence of finitely generated algebras. It is acted on by the trivial spectral sequence $E(\mathscr{A})$ whose pages equal $H^*(G, \mathscr{A})$. But we have proved that E_1^{**} is Noetherian over $H^*(G, \mathscr{A})$, so by the usual trick (see [6], [9], or [24, Lemma 3.9]), the spectral sequence $E(A)$ stops, which proves that $H^*(G, A)$ is finitely generated. □

4. Proof of Theorem 3.1

By [22, Proposition 3.21], the divided powers Γ^ℓ admit a twist-compatible coresolution J_ℓ. So by [22, Proposition 3.18], we have a bicomplex $A(J_\ell)$ whose totalization yields an $H^*_{\mathscr{P}}$-acyclic coresolution of $\Gamma^\ell(\mathrm{gl}^{(1)})$. In particular, the homology of the totalization of $H^0_{\mathscr{P}}(A(J_\ell))$ computes $H^*_{\mathscr{P}}(\Gamma^\ell(\mathrm{gl}^{(1)}))$.

The plan of the proof of Theorem 3.1 is the following. First, we build the maps ψ_ℓ. To be more specific, we build maps ϑ_ℓ which send each element of degree d of $\Gamma^\ell(H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}))$ to a homogeneous cocycle of bidegree $(0,d)$ in the bicomplex $H^0_{\mathscr{P}}(A(J_\ell))$. Our maps ψ_ℓ can then be induced by the ϑ_ℓ.

Second, we show the relations between the classes on the cochain level. In this step, we encounter the following problem: the cup product of two classes is represented by a cocycle in the bicomplex $H^0_{\mathscr{P}}(A(J_\ell)\otimes A(J_m))$, while we want to have it represented by a cocycle in $H^0_{\mathscr{P}}(A(J_\ell\otimes J_m))$. So we have to investigate further the compatibility of the functor A with cup products.

Finally, we prove Theorem 3.1(2) by reducing to one-parameter subgroups.

Notation and sign conventions 4.1

If $\mathfrak{A}$ is an additive category, we denote by $\mathrm{Ch}^{\geq 0}(\mathfrak{A})$ (resp., p-$\mathrm{Ch}^{\geq 0}(\mathfrak{A})$; resp., bi-$\mathrm{Ch}^{\geq 0}(\mathfrak{A})$) the category of nonnegative cochain complexes (resp., p-complexes; resp., bicomplexes) in $\mathfrak{A}$.

If $\mathfrak{A}$ is equipped with a tensor product, then $\mathrm{Ch}^{\geq 0}(\mathfrak{A})$ inherits a tensor product. The differential of the tensor product $C\otimes D$ involves a Koszul sign: the restriction of $d_{C\otimes D}$ to $C^i\otimes D^j$ equals $d_C\otimes\mathrm{Id}+(-1)^i\mathrm{Id}\otimes d_D$. The category p-$\mathrm{Ch}^{\geq 0}(\mathfrak{A})$ also inherits a tensor product, but the p-differential of $C\otimes D$ does not involve any sign: $d_{C\otimes D}=d_C\otimes\mathrm{Id}+\mathrm{Id}\otimes d_D$.

Now we turn to bicomplexes. First, we may view a complex $C^\bullet$ whose terms C^j are chain complexes as a bicomplex $C^{\bullet,\bullet}$ whose object $C^{i,j}$ is the ith object of the complex C^j (i.e., the complexes C^j are the rows of $C^{\bullet,\bullet}$). Thus, we obtain an identification:

$$\mathrm{Ch}^{\geq 0}\big(\mathrm{Ch}^{\geq 0}(\mathfrak{A})\big)=\text{bi-}\mathrm{Ch}^{\geq 0}(\mathfrak{A}).$$

Being a category of cochain complexes, the term on the left-hand side has a tensor product. If C is a bicomplex, let us denote by $d_C^{i,j}:C^{i,j}\to C^{i+1,j}$ its first differential, and by $\partial_C^{i,j}:C^{i,j}\to C^{i,j+1}$ its second one. Then one checks that the tensor product on bicomplexes induced by the identification is such that the restriction of $d_{C\otimes D}$ (resp., $\partial_{C\otimes D}$) to $C^{i_1,j_1}\otimes D^{i_2,j_2}$ equals $d_C\otimes\mathrm{Id}+(-1)^{i_1}\mathrm{Id}\otimes d_D$ (resp., $\partial_C\otimes\mathrm{Id}+(-1)^{j_1}\mathrm{Id}\otimes\partial_D$).

We define the totalization $\mathrm{Tot}(C)$ of a bicomplex C with the Koszul sign convention: the restriction of $d_{\mathrm{Tot}(C)}$ to $C^{i,j}$ equals $d_C+(-1)^i\partial_C$. If C, D are two bicomplexes, there is a canonical isomorphism of complexes, $\mathrm{Tot}(C)\otimes\mathrm{Tot}(D)\simeq\mathrm{Tot}(C\otimes D)$, which sends an element $x\otimes y\in C^{i_1,j_1}\otimes D^{i_2,j_2}$ to $(-1)^{j_1i_2}x\otimes y$.

4.1. Construction of the ψ_ℓ, $\ell \geq 1$

Let ℓ be a positive integer. By [22, Propositions 3.18, 3.21], we have a bicomplex $H^0_{\mathscr{P}}(A(J_\ell))$ whose homology computes the cohomology of the bifunctor $\Gamma^\ell(\mathrm{gl}^{(1)})$. We now recall the description of the first two columns of this bicomplex. As in [22, Section 4], we denote by A_1 the p-coresolution of $\mathrm{gl}^{(1)}$ obtained by precomposing the p-complex $T(S^1)$ by the bifunctor gl. The symmetric group $\mathfrak{S}_\ell$ acts on the p-complex $A_1^{\otimes \ell}$ by permuting the factors of the tensor product. (Unlike the case of ordinary complexes, the action of $\mathfrak{S}_\ell$ does not involve a Koszul sign since the tensor product of p-complexes does not involve any sign.) Contracting the p-complex $A_1^{\otimes \ell}$ and applying $H^0_{\mathscr{P}}$, we obtain an action of $\mathfrak{S}_\ell$ on the ordinary complex $H^0_{\mathscr{P}}((A_1^{\otimes \ell})_{[1]})$. By [22, Lemma 4.2], the first two columns $H^*_{\mathscr{P}}(A(J_\ell)^{0,\bullet}) \to H^*_{\mathscr{P}}(A(J_\ell)^{1,\bullet})$ of $H^0_{\mathscr{P}}(A(J_\ell))$ equal

$$\underbrace{H^0_{\mathscr{P}}\left((A_1^{\otimes \ell})_{[1]}\right)}_{\text{column of index 0}} \xrightarrow{\prod(1-\tau_i)} \underbrace{\bigoplus_{i=0}^{\ell-2} H^0_{\mathscr{P}}\left((A_1^{\otimes \ell})_{[1]}\right)}_{\text{column of index 1}},$$

where $\tau_i \in \mathfrak{S}_\ell$ is the transposition which exchanges $i+1$ and $i+2$ (and with the convention that the second column is null if $\ell = 1$). Thus we have the following.

LEMMA 4.2

Let $\mathscr{Z}_\ell^{\text{even}}$ be the set of homogeneous cocycles of bidegree $(0, d)$, d even, in the bicomplex $H^0_{\mathscr{P}}(A(J_\ell))$. Then $\mathscr{Z}_\ell^{\text{even}}$ identifies as the set of even-degree cocycles of the complex $H^0_{\mathscr{P}}((A_1^{\otimes \ell})_{[1]})$, which are invariant under the action of $\mathfrak{S}_\ell$.

Now we turn to building a map $\vartheta_\ell : \Gamma^\ell(H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})) \to \mathscr{Z}_\ell^{\text{even}}$. In view of Lemma 4.2, it suffices to build a $\mathfrak{S}_\ell$-equivariant map $\vartheta_\ell : H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})^{\otimes \ell} \to H^0_{\mathscr{P}}((A_1^{\otimes \ell})_{[1]})$.

Let us first recall what we know about $H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$. By [8, Theorem 1.5] and [9, Theorem 4.5], the graded vector space $H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$ is concentrated in degrees $2i$, $0 \leq i < p$, and is one-dimensional in these degrees. Following [21], we denote by $e_1(i)$ a generator of degree $2i$ of this graded vector space. The homology of the complex $H^0_{\mathscr{P}}(A_{1[1]})$ computes the cohomology of the bifunctor $\mathrm{gl}^{(1)}$. Thus we may choose for each integer i, $0 \leq i < p$, a cycle z_i representing the cohomology class $e_1(i)$ in this complex. The cycles z_i determine a graded map $H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \to H^0_{\mathscr{P}}(A_{1[1]})$. By [22, Proposition 3.3], we may take cup products on the cochain level to obtain for each $\ell \geq 1$ a map

$$H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})^{\otimes \ell} \to H^0_{\mathscr{P}}(A_{1[1]})^{\otimes \ell} \xrightarrow{\cup} H^0_{\mathscr{P}}\left((A_{1[1]})^{\otimes \ell}\right).$$

Moreover, we define chain maps $h_\ell : (A_{1[1]})^{\otimes \ell} \to (A_1^{\otimes \ell})_{[1]}$ by iterated use of [22, Proposition 2.7]. More specifically, h_1 is the identity, and $h_\ell = h_{A_1^{\otimes \ell-1}, A_1} \circ (h_{\ell-1} \otimes h_1)$.

LEMMA 4.3
Let ℓ be a positive integer, and let ϑ_ℓ be the composite

$$\vartheta_\ell := H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})^{\otimes \ell} \to H^0_{\mathscr{P}}\big((A_{1[1]})^{\otimes \ell}\big) \xrightarrow{H^0_{\mathscr{P}}(h_\ell)} H^0_{\mathscr{P}}\big((A_1^{\otimes \ell})_{[1]}\big).$$

Then ϑ_ℓ satisfies the following two properties.

(1) *The image of ϑ_ℓ is contained in the set of even-degree cocycles of $H^0_{\mathscr{P}}((A_1^{\otimes \ell})_{[1]})$.*

(2) *The map ϑ_ℓ is $\mathfrak{S}_\ell$-equivariant.*

Proof
The first property is straightforward from the definition of ϑ_ℓ. We prove the second one. The map $H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})^{\otimes \ell} \to H^0_{\mathscr{P}}((A_{1[1]})^{\otimes \ell})$ is defined using cup products; hence it is $\mathfrak{S}_\ell$-equivariant. Thus, to prove the lemma, we have to study the map $h_\ell : (A_{1[1]})^{\otimes \ell} \to (A_1^{\otimes \ell})_{[1]}$.

Recall that h_ℓ is built by iterated uses of [22, Proposition 2.7]. Thus, if we define the graded object $p(A_1, \dots, A_1) = \bigoplus_{i_1,\dots,i_\ell} \bigotimes_{s=1}^{\ell} A_1^{i_s p}$ with the component $\bigotimes_{s=1}^{\ell} A_1^{i_s p}$ in degree $2(\sum i_s)$, we have well-defined inclusions of $p(A_1, \dots, A_1)$ into the complexes $(A_{1[1]})^{\otimes \ell}$ and $(A_1^{\otimes \ell})_{[1]}$. Moreover, h_ℓ fits into a commutative diagram:

$$\begin{array}{ccc}
(A_{1[1]})^{\otimes \ell} & \xrightarrow{\quad h_\ell \quad} & (A_1^{\otimes \ell})_{[1]} \\
{\scriptstyle (a)}\big\uparrow & & \big\uparrow{\scriptstyle (b)} \\
p(A_1, \dots, A_1) & = & p(A_1, \dots, A_1)
\end{array}$$

Let $\mathfrak{S}_\ell$ act on $p(A_1, \dots, A_1)$ by permuting the factors of the tensor product, let it act on $(A_{1[1]})^{\otimes \ell}$ by permuting the factors of the tensor product with a Koszul sign, and let it act on $(A_1^{\otimes \ell})_{[1]}$ by permuting the factors of the tensor product $A_1^{\otimes \ell}$ (without sign). Then the map (b) is equivariant, and the map (a) is also equivariant since $p(A_1, \dots, A_1)$ is concentrated in even degrees. The map h_ℓ is *not* equivariant. However, by definition the equivariant map $H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})^{\otimes \ell} \to H^0_{\mathscr{P}}((A_{1[1]})^{\otimes \ell})$ factors through $H^0_{\mathscr{P}}(p(A_1, \dots, A_1))$ so that postcomposition of this map by $H^0_{\mathscr{P}}(h_\ell)$ (i.e., the map ϑ_ℓ) is in fact equivariant. □

Notation 4.4
By Lemmas 4.2 and 4.3, for all $\ell \geq 1$, the map ϑ_ℓ induces a map $\Gamma^\ell(H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})) \to \mathscr{Z}^{\mathrm{even}}_\ell$. We denote by ψ_ℓ the composite

$$\psi_\ell := \Gamma^\ell\big(H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})\big) \to \mathscr{Z}^{\mathrm{even}}_\ell \to H^*_{\mathscr{P}}\big(\Gamma^\ell(\mathrm{gl}^{(1)})\big).$$

LEMMA 4.5
The map ψ_1 equals the identity map.

Proof
For $\ell = 1$, ϑ_1 is just the map $H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \to \mathrm{Hom}(\Gamma^p(\mathrm{gl}), A_{1[1]})$ which sends the generator $e_1(i)$ of $H^{2i}_{\mathscr{P}}(\mathrm{gl}^{(1)})$ to the cycle z_i representing this generator. Moreover, by definition of z_i, the map $\mathscr{Z}^{\mathrm{even}}_1 \twoheadrightarrow H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$ sends z_i to $e_1(i)$. Thus, for all i, ψ_1 sends $e_1(i)$ to itself. □

4.2. Proof of Theorem 3.1(3)
Let $\mathscr{P}_k$ be the strict polynomial functor category, and let $\mathscr{T}\mathscr{P}_k$ be the twist-compatible subcategory (see [22, Definition 3.9]). Before proving Theorem 3.1(3), we need to study further properties of the functor $A : \mathrm{Ch}^{\geq 0}(\mathscr{T}\mathscr{P}_k) \to \text{bi-}\mathrm{Ch}^{\geq 0}(\mathscr{P}_k(1,1))$ (see [22, Definition 3.17]). Recall that A is defined as the composite of the following three functors:

(1) the Troesch coresolution functor (see [22, Proposition 3.13])

$$T : \mathrm{Ch}^{\geq 0}(\mathscr{T}\mathscr{P}_k) \to p\text{-}\mathrm{Ch}^{\geq 0}\big(\mathrm{Ch}^{\geq 0}(\mathscr{P}_k)\big);$$

(2) the contraction functor

$$-_{[1]} : p\text{-}\mathrm{Ch}^{\geq 0}\big(\mathrm{Ch}^{\geq 0}(\mathscr{P}_k)\big) \to \mathrm{Ch}^{\geq 0}\big(\mathrm{Ch}^{\geq 0}(\mathscr{P}_k)\big);$$

(3) precomposition by the bifunctor gl

$$-\circ \mathrm{gl} : \mathrm{Ch}^{\geq 0}\big(\mathrm{Ch}^{\geq 0}(\mathscr{P}_k)\big) \to \mathrm{Ch}^{\geq 0}\big(\mathrm{Ch}^{\geq 0}(\mathscr{P}_k(1,1))\big) = \text{bi-}\mathrm{Ch}^{\geq 0}\big(\mathscr{P}_k(1,1)\big).$$

All the categories coming into play in the definition of A are equipped with tensor products (see Notation and sign conventions 4.1). The functors T and $-\circ\mathrm{gl}$ commute with tensor products, but $-_{[1]}$ does not. As a result, if F, G are homogeneous strict polynomial functors of respective degree f, g and with respective twist-compatible coresolutions J_F, J_G, we have two (in general nonisomorphic) $H^*_{\mathscr{P}}$-acyclic coresolutions of the tensor product $F \otimes G$ at our disposal:

$$\mathrm{Tot}\big(A(J_F)\big) \otimes \mathrm{Tot}\big(A(J_G)\big) \qquad \text{and} \qquad \mathrm{Tot}\big(A(J_F \otimes J_G)\big).$$

Now the problem is the following. On the one hand, cycles representing cup products of classes in the cohomology of F and G are easily identified using the first complex. Indeed, by [22, Proposition 3.3], the cup product

$$H^*_{\mathscr{P}}\big(F(\mathrm{gl}^{(1)})\big) \otimes H^*_{\mathscr{P}}\big(G(\mathrm{gl}^{(1)})\big) \to H^*_{\mathscr{P}}\big(F(\mathrm{gl}^{(1)}) \otimes G(\mathrm{gl}^{(1)})\big)$$

is defined at the cochain level by sending cocycles x and y, respectively, in $\mathrm{Hom}\big(\Gamma^{pf}(\mathrm{gl}), \mathrm{Tot}(A(J_F))\big)$ and $\mathrm{Hom}\,\big(\Gamma^{pg}(\mathrm{gl}), \mathrm{Tot}(A(J_G))\big)$ to the cocycle

$$x \cup y := (x \otimes y) \circ \Delta_{pf,pg} \in \mathrm{Hom}\big(\Gamma^{p(f+g)}(\mathrm{gl}), \mathrm{Tot}(A(J_F)) \otimes \mathrm{Tot}(A(J_G))\big),$$

where $\Delta_{pf,pg}$ is the diagonal map $\Gamma^{p(f+g)}(\mathrm{gl}) \to \Gamma^{pf}(\mathrm{gl}) \otimes \Gamma^{pg}(\mathrm{gl})$. But on the other hand, by functoriality of A, if $E \in \mathscr{P}_k$, then the effect of a morphism $E \to F \otimes G$ is easily computed in $H^0_{\mathscr{P}}\big(\mathrm{Tot}(A(J_F \otimes J_G))\big)$. So we want to be able to identify cup products in $H^0_{\mathscr{P}}\big(\mathrm{Tot}(A(J_F \otimes J_G))\big)$ rather than in $H^0_{\mathscr{P}}\big(\mathrm{Tot}(A(J_F)) \otimes \mathrm{Tot}(A(J_G))\big)$. This is the purpose of Lemma 4.6.

LEMMA 4.6

Let F, G be homogeneous strict polynomial functors of degree f, g which admit twist-compatible coresolutions J_F and J_G. Let i, j, ℓ, m be nonnegative integers, and let

$$x_{i,2j} \in \mathrm{Hom}_{\mathscr{P}^{pf}_{pf}}\left(\Gamma^{pf}(\mathrm{gl}), A(J_F)\right), \; y_{\ell,2m} \in \mathrm{Hom}_{\mathscr{P}^{pg}_{pg}}\left(\Gamma^{pg}(\mathrm{gl}), A(J_G)\right)$$

be homogeneous cocycles of respective bidegrees $(i, 2j)$ and $(\ell, 2m)$.

(1) *The object $A(J_F)^{i,2j} \otimes A(J_G)^{\ell,2m}$ appears once and only once in the bicomplex $A(J_F \otimes J_G)$. It appears in bidegree $(i+\ell, 2j+2m)$. In particular, the formula*

$$(x_{i,2j} \otimes y_{\ell,2m}) \circ \Delta_{pf,pg} \in \mathrm{Hom}\big(\Gamma^{p(f+g)}(\mathrm{gl}), A(J_F)^{i,2j} \otimes A(J_G)^{\ell,2m}\big)$$

defines a homogeneous element of bidegree $(i+\ell, 2j+2m)$ in the bicomplex $H^0_{\mathscr{P}}(A(J_F \otimes J_G))$.

(2) *The element $(x_{i,2j} \otimes y_{\ell,2m}) \circ \Delta_{pf,pg}$ is actually a cocycle and represents the cup product $[x_{i,2j}] \cup [y_{\ell,2m}]$ in $H^0_{\mathscr{P}}\big(\mathrm{Tot}(A(J_F \otimes J_G))\big)$.*

Proof

Since T commutes with tensor products (see [22, Proposition 3.13]), the bicomplex $A(J_F \otimes J_G)$ is naturally isomorphic to the precomposition by gl of the bicomplex $(T(J_F) \otimes T(J_G))_{[1]}$, while $A(J_F) \otimes A(J_G)$ equals the precomposition by gl of the bicomplex $T(J_F)_{[1]} \otimes T(J_G)_{[1]}$. Recall that in the identification of $\mathrm{Ch}^{\geq 0}\big(\mathrm{Ch}^{\geq 0}(\mathscr{P}_k(1,1))\big)$ and bi-$\mathrm{Ch}^{\geq 0}(\mathscr{P}_k(1,1))$, the jth object of a complex of complexes $C^\bullet$ corresponds to the jth row of the bicomplex $C^{\bullet,\bullet}$ (i.e., the elements of bidegree $(*, j)$). So the first statement simply follows from [22, Lemma 2.2]. Furthermore, by [22, Proposition

2.4] there is a map of bicomplexes

$$A(J_F)\otimes A(J_G)\to A(J_F\otimes J_G)$$

which is the identity on $A(J_F)^{i,2j}\otimes A(J_G)^{\ell,2m}$. Applying the functor Tot, we obtain a map of $H^*_{\mathscr{P}}$-acyclic coresolutions

$$\theta\ :\ \mathrm{Tot}\big(A(J_F)\big)\otimes\mathrm{Tot}\big(A(J_G)\big)\simeq\mathrm{Tot}\big(A(J_F)\otimes A(J_G)\big)\to\mathrm{Tot}\big(A(J_F\otimes J_G)\big)$$

over the identity map of $F(\mathrm{gl}^{(1)})\otimes G(\mathrm{gl}^{(1)})$, and whose restriction to $A(J_F)^{i,2j}\otimes A(J_G)^{\ell,2m}$ equals the identity. (More specifically, this equality holds up to a $(-1)^{2j\ell}=1$ sign coming from the sign in the formula $\mathrm{Tot}(C\otimes D)\simeq\mathrm{Tot}(C)\otimes\mathrm{Tot}(D)$.)

By definition of the cup product, $(x_{i,2j}\otimes y_{\ell,2m})\circ\Delta_{pf,pg}$ is a cocycle representing $[x_{i,2j}]\cup[y_{\ell,2m}]$ in $H^0_{\mathscr{P}}\big(\mathrm{Tot}(A(J_F))\otimes\mathrm{Tot}(A(J_G))\big)$. Applying θ, we obtain that $(x_{i,2j}\otimes y_{\ell,2m})\circ\Delta_{pf,pg}$ is a cocycle in $H^0_{\mathscr{P}}\big(\mathrm{Tot}(A(J_F\otimes J_G))\big)$, representing the same cup product. □

We now turn to the specific situation of Theorem 3.1(3), that is, $F=\Gamma^\ell$ and $G=\Gamma^m$. We first determine explicit maps between the bicomplexes $A(J_\ell\otimes J_m)$ and $A(J_{\ell+m})$, which lift the multiplication $\Gamma^\ell(\mathrm{gl}^{(1)})\otimes\Gamma^m(\mathrm{gl}^{(1)})\to\Gamma^{\ell+m}(\mathrm{gl}^{(1)})$ and the diagonal $\Gamma^{\ell+m}(\mathrm{gl}^{(1)})\to\Gamma^\ell(\mathrm{gl}^{(1)})\otimes\Gamma^m(\mathrm{gl}^{(1)})$. To do this, we first need new information about the twist-compatible coresolutions J_ℓ from [22, Proposition 3.21].

LEMMA 4.7

Let ℓ,m be positive integers.

(1) *The multiplication $\Gamma^\ell\otimes\Gamma^m\to\Gamma^{\ell+m}$ lifts to a twist-compatible chain map $J_\ell\otimes J_m\to J_{\ell+m}$. This chain map is given in degree zero by the shuffle product $(\bigotimes^\ell)\otimes(\bigotimes^m)=J^0_\ell\otimes J^0_m\to J^0_{\ell+m}=\bigotimes^{\ell+m}$, which sends a tensor $\bigotimes_{i=1}^{m+\ell}x_i$ to the sum $\sum_{\sigma\in\mathrm{Sh}(\ell,m)}\otimes_{i=1}^{m+\ell}x_{\sigma^{-1}(i)}$.*

(2) *The diagonal $\Gamma^{\ell+m}\to\Gamma^\ell\otimes\Gamma^m$ lifts to a twist-compatible chain map $J_{\ell+m}\to J_\ell\otimes J_m$. This chain map equals the identity map in degree zero.*

Proof

The reduced bar construction yields a functor from the category of commutative differential graded augmented algebras over k to the category of commutative differential graded bialgebras over k (see [13] (resp., [7]) for the algebra (resp., coalgebra) structure). (This bialgebra structure is actually a Hopf algebra structure, but we do not need this fact.)

The category of strict polynomial functors splits as a direct sum of subcategories of homogeneous functors. Taking the $(m+\ell)$ polynomial degree part of the multiplication (resp., comultiplication) of $\overline{B}\big(\overline{B}(S^*(-))\big)$, we obtain chain maps $\bigoplus J^\bullet_i\otimes J^\bullet_j\to J^\bullet_{\ell+m}$

and $J^\bullet_{\ell+m} \to \bigoplus J^\bullet_i \otimes J^\bullet_j$. (The sums are taken over all nonnegative integers i, j such that $i + j = \ell + m$.) The bialgebra structure of $\overline{B}\big(\overline{B}(S^*(-))\big)$ is defined using only the *algebra* structure of S^*. But the multiplication of S^* is a twist-compatible map, and the twist-compatible category is additive and stable under tensor products (see [22, Lemmas 3.8, 3.10]). So the chain maps are twist-compatible.

Next, we identify the chain maps in degree zero. We begin with the map $J^0_\ell \otimes J^0_m \to J^0_{\ell+m}$ induced by the multiplication of the bar construction. By [22, Lemma 3.18], for all $i \geq 1$ we have $J^0_i = \overline{B}_1(S^*(-))^{\bigotimes i} = \bigotimes^i$. The product $\overline{B}\big(\overline{B}(S^*(-))\big)^{\otimes 2} \to \overline{B}\big(\overline{B}(S^*(-))\big)$ is given by the shuffle product formula in [13, page 313]; more precisely, it sends the tensor $\bigotimes_{i=1}^{m+\ell} x_i$ to the sum $\sum_{\sigma\in \mathrm{Sh}(\ell,m)} \otimes_{i=1}^{m+\ell} x_{\sigma^{-1}(i)}$. The signs in this shuffle product are all positive since the x_i are elements of degree $1 + 1 = 2$ in the chain complex $\overline{B}_\bullet\big(\overline{B}(S^*(-))\big)$. The identification of the map $J^0_{m+\ell} \to J^0_m \otimes J^0_\ell$ induced by the diagonal is simpler. The coproduct in $\overline{B}\big(\overline{B}(S^*(-))\big)$ is given by the deconcatenation formula [7, page 268]: $\Delta[x_1|\cdots|x_{\ell+m}] = \sum_{i=0}^{m+\ell} [x_1|\cdots|x_i]\otimes[x_{i+1}|\cdots|x_{m+\ell}]$. Thus, the map $J^0_{m+\ell} \to J^0_m \otimes J^0_\ell$ sends the tensor product $\bigotimes_{i=1}^{m+\ell} x_i$ to itself.

Finally, with the description of the chain maps in degree zero, one easily checks that $J_\ell \otimes J_m \to J_{\ell+m}$ (resp., $J_{\ell+m} \to J_\ell \otimes J_m$) lifts the multiplication $\Gamma^\ell \otimes \Gamma^m \to \Gamma^{\ell+m}$ (resp., the comultiplication $\Gamma^{\ell+m} \to \Gamma^\ell \otimes \Gamma^m$). (In fact, this actually proves that the quasi isomorphism $\Gamma^* \to \overline{B}\big(\overline{B}(S^*(-))\big)$ is a Hopf algebra morphism.) □

Applying the functor A, we obtain the following.

LEMMA 4.8

Let ℓ, m be positive integers.

(1) *The multiplication $\Gamma^\ell(\mathrm{gl}^{(1)}) \otimes \Gamma^m(\mathrm{gl}^{(1)}) \to \Gamma^{\ell+m}(\mathrm{gl}^{(1)})$ lifts to a map of bicomplexes $A(J_\ell \otimes J_m) \to A(J_{\ell+m})$. The restriction of this map to the columns of index zero equals*

$$A(J_\ell \otimes J_m)^{0,\bullet} = (A_1^{\otimes \ell+m})_{[1]} \xrightarrow{\mathrm{sh}_{[1]}} (A_1^{\otimes \ell+m})_{[1]} = A(J_{\ell+m})^{0,\bullet},$$

where sh *is the unsigned shuffle map, which sends a tensor $\bigotimes_{i=1}^{m+\ell} x_i$ to the sum $\sum_{\sigma\in \mathrm{Sh}(\ell,m)} \otimes_{i=1}^{m+\ell} x_{\sigma^{-1}(i)}$.*

(2) *The diagonal $\Gamma^{\ell+m}(\mathrm{gl}^{(1)}) \to \Gamma^\ell(\mathrm{gl}^{(1)}) \otimes \Gamma^m(\mathrm{gl}^{(1)})$ lifts to a twist-compatible chain map $A(J_{\ell+m}) \to A(J_\ell \otimes J_m)$. The restriction of this map to the columns of index zero equals the identity map of $(A_1^{\otimes \ell+m})_{[1]}$.*

Next, we identify cycles representing the cup products $\psi_\ell(x)\cup\psi_m(y)$ in the bicomplex $H^0_{\mathscr{P}}(A(J_\ell \otimes J_m))$.

LEMMA 4.9
Let $x \in \Gamma^\ell H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$ and $y \in \Gamma^m H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$ be classes of homogeneous degrees $2d$ and $2e$. Then $\vartheta_{\ell+m}(x \otimes y)$ is a cocycle of bidegree $(0, 2d)$ in the bicomplex $H^0_{\mathscr{P}}(A(J_\ell \otimes J_m))$. Moreover, it represents the cup product $\psi_\ell(x) \cup \psi_m(y) \in H^*_{\mathscr{P}}(\Gamma^\ell(\mathrm{gl}^{(1)}) \otimes \Gamma^m(\mathrm{gl}^{(1)}))$.

Proof
By definition, $\psi_\ell(x)$ is represented by the homogeneous cocycle $\vartheta_\ell(x)$ of bidegree $(0, 2d)$ in the bicomplex $\mathrm{Hom}(\Gamma^{p\ell}(\mathrm{gl}), A(J_\ell))$ (and similarly for $\psi_m(y)$). Then, by Lemma 4.6, $\psi_\ell(x) \cup \psi_m(y)$ is represented by the cocycle $(\vartheta_\ell(x) \otimes \vartheta_m(y)) \circ \Delta_{\ell p, mp}$ in the bicomplex $\mathrm{Hom}(\Gamma^{p(\ell+m)}(\mathrm{gl}), A(J_\ell \otimes J_m))$. Now, if $x = \bigotimes_{s=1}^{\ell}(e(i_s))$ and $y = \bigotimes_{s=\ell+1}^{m}(e(i_s))$, we compute that $\vartheta_{\ell+m}(x \otimes y)$ and $(\vartheta_\ell(x) \otimes \vartheta_m(y)) \circ \Delta_{\ell p, mp}$ both equal the element $(\bigotimes_{i=1}^{\ell+m} z_i) \circ \Delta_{p,\dots,p}$, where $\Delta_{p,\dots,p}$ is the diagonal $\Gamma^{p(\ell+m)}(\mathrm{gl}) \to \Gamma^p(\mathrm{gl})^{\otimes \ell+m}$. □

We are now ready to prove Theorem 3.1(3). We begin with the commutativity of the diagram involving the multiplication. Let $x \in \Gamma^\ell H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$ and $y \in \Gamma^m H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$ be homogeneous elements of respective degrees $2d$ and $2e$. By Lemmas 4.8 and 4.9, $m_{\ell,m\,*}(\psi_\ell(x) \cup \psi_m(y))$ is represented by the cocycle

$$\sum_{\sigma \in \mathrm{Sh}(\ell,m)} \sigma.\vartheta_{\ell+m}(x \otimes y)$$

of bidegree $(0, 2d + 2e)$ in the bicomplex $H^0_{\mathscr{P}}(A(J_{\ell+m}))$. By definition of $\psi_{\ell+m}$, $\psi_{\ell+m}(m_{\ell,m}(x \otimes y))$ is represented by the cocycle

$$\vartheta_{\ell+m}\Big(\sum_{\sigma \in \mathrm{Sh}(\ell,m)} \sigma.(x \otimes y)\Big)$$

in the same bicomplex. Since $\vartheta_{\ell+m}$ is equivariant (see Lemma 4.3), these two cocycles are equal. Hence, the diagram involving the multiplication is commutative. The diagram involving the comultiplication commutes for a similar reason: if $x \in \Gamma^{\ell+m} H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$, the cohomology classes $(\psi_\ell \cup \psi_m)(\Delta_{\ell,m}(x))$ and $\Delta_{\ell,m\,*}(\psi_{\ell+m}(x))$ are both represented by the cycle $\vartheta_{\ell+m}(x)$. This concludes the proof of Theorem 3.1(3).

4.3. Proof of Theorem 3.1(2)
To prove Theorem 3.1(2), it suffices to prove, for all $n \geq p$, the injectivity of the composite

$$\Gamma^\ell H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \xrightarrow{\psi_\ell} H^*_{\mathscr{P}}\big(\Gamma^\ell(\mathrm{gl}^{(1)})\big) \xrightarrow{\phi_{\Gamma^\ell(\mathrm{gl}^{(1)}),n}} H^*\big(\mathrm{GL}_{n,k}, \Gamma^\ell(\mathfrak{gl}_n^{(1)})\big)$$
$$\xrightarrow{\Delta_{1,\dots,1\,*}} H^*(\mathrm{GL}_{n,k}, \mathfrak{gl}_n^{(1)\otimes\ell}).$$

By naturality of the maps $\phi_{\Gamma^\ell(\mathrm{gl}^{(1)}),n}$ (see [22, Theorem 1.3]) and by the compatibility of the ϕ_i with diagonals and cup products given in Theorem 3.1(3), this composite equals the composite

$$\Gamma^\ell H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \hookrightarrow H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})^{\otimes \ell} \to H^*(\mathrm{GL}_{n,k}, \mathfrak{gl}_n^{(1)})^{\otimes \ell} \xrightarrow{\cup} H^*(\mathrm{GL}_{n,k}, \mathfrak{gl}_n^{(1)\otimes \ell}).$$

Thus, the proof of Theorem 3.1(2) follows from the following.

LEMMA 4.10
Let k be a field of characteristic $p > 0$, and let $j \geq 1$ be an integer. For all $n \geq p$, the following map is injective:

$$\begin{array}{rcl} \bigcup_{i=1}^j \phi_{\mathrm{gl}^{(1)},n} : H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})^{\otimes j} & \to & H^*(\mathrm{GL}_{n,k}, (\mathfrak{gl}_n^{(1)})^{\otimes j}) \\ \bigotimes_{i=1}^j c_i & \mapsto & \bigcup_{i=1}^j \phi_{\mathrm{gl}^{(1)},n}(c_i) \end{array}$$

Proof
We prove this lemma by reducing our cohomology classes to an infinitesimal one-parameter subgroup $\mathbb{G}_{a1}$ of $\mathrm{GL}_{n,k}$, as is done in [21]. Since $n \geq p$, we can find a p-nilpotent matrix $\alpha \in \mathfrak{gl}_n$. Using this matrix, we define an embedding $\mathbb{G}_{a1} \to \mathbb{G}_a \xrightarrow{\exp_\alpha} \mathrm{GL}_{n,k}$. For all ℓ, this embedding makes the $\mathrm{GL}_{n,k}$-module $\mathfrak{gl}_n^{(1)\otimes\ell}$ into a trivial $\mathbb{G}_{a1}$-module. Thus, there is an isomorphism $H^*(\mathbb{G}_{a1}, \mathfrak{gl}_n^{(1)\otimes\ell}) \simeq H^*(\mathbb{G}_{a1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes\ell}$. The algebra $H^*(\mathbb{G}_{a1}, k)$ is computed in [5]. In particular, $H^{\mathrm{even}}(\mathbb{G}_{a1}, k) = k[x_1]$ is a polynomial algebra on one generator x_1 of degree 2. Let us thrash out the compatibility of this isomorphism with the cup product. If $x_1^\ell \otimes \beta_\ell$ and $x_1^m \otimes \beta_m$ are classes in $H^*(\mathbb{G}_{a1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes\ell}$ (resp., $H^*(\mathbb{G}_{a1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes m}$), their cup product is the class $x_1^{\ell+m} \otimes (\beta_\ell \otimes \beta_m)$ in $H^*(\mathbb{G}_{a1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes\ell+m}$.

We recall that $H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})$ is a graded module with basis the classes $e_1(i)$ of degree $2i$ for $0 \leq i < p$. By [21, Theorem 4.9], the composite

$$H^*_{\mathscr{P}}(\mathrm{gl}^{(1)}) \underset{\simeq}{\xrightarrow{\phi_{\mathrm{gl}(1),n}}} H^*(\mathrm{GL}_{n,k}, \mathfrak{gl}_n^{(1)}) \to H^*(\mathbb{G}_{a1}, \mathfrak{gl}_n^{(1)}) \simeq H^*(\mathbb{G}_{a1}, k) \otimes \mathfrak{gl}_n^{(1)}$$

sends $e_1(i)$ to the class $x_1^i \otimes (\alpha^{(1)})^i \in H^*(\mathbb{G}_{a1}, k) \otimes \mathfrak{gl}_n^{(1)}$. Since restriction to $\mathbb{G}_{a1}$ is compatible with cup products, the composite

$$H^*_{\mathscr{P}}(\mathrm{gl}^{(1)})^{\otimes j} \to H^*(\mathrm{GL}_{n,k}, \mathfrak{gl}_n^{(1)\otimes j}) \to H^*(\mathbb{G}_{a1}, k) \otimes \mathfrak{gl}_n^{(1)\otimes j}$$

sends the tensor product $\bigotimes_{\ell=1}^j e(i_\ell)$ to the class $x_1^{\sum_{\ell=1}^j i_\ell} \otimes (\bigotimes_{\ell=1}^j (\alpha^{(1)})^{i_\ell})$. As a result, this composite sends the basis $\big(\bigotimes_{i=\ell}^j e(i_\ell)\big)_{(i_1,\dots,i_\ell)}$ into the linearly independent family $\big(x_1^m \otimes (\bigotimes_{\ell=1}^j (\alpha^{(1)})^{i_\ell})\big)_{(m,i_1,\dots,i_\ell)}$. Hence the map $\bigcup_{i=1}^j \phi_{\mathrm{gl}^{(1)},n}$ is injective. □

Part II. The second proof

5. The starting point

Most notation is the same as in [23]. We work over a field k of positive characteristic p. Fix an integer n with $n > 1$. The important case is when n is large. So if one finds this convenient, one may take $n \geq p$. We wish to draw conclusions from the following result.

THEOREM 5.1 (Lifted universal cohomology classes; see [22])
There are cohomology classes $c[m]$ so that
(1) $c[1] \in H^2(\mathrm{GL}_{n,k}, \mathfrak{gl}_n^{(1)})$ *is nonzero;*
(2) *for $m \geq 1$, the class $c[m] \in H^{2m}(\mathrm{GL}_{n,k}, \Gamma^m(\mathfrak{gl}_n^{(1)}))$ lifts $c[1] \cup \cdots \cup c[1] \in H^{2m}(\mathrm{GL}_{n,k}, \bigotimes^m(\mathfrak{gl}_n^{(1)}))$.*

Remark 5.2
What we really have in mind is that $c[1]$ is the Witt vector class of [23, Section 4], which is certainly nonzero. The computation of $H^2(\mathrm{GL}_{n,k}, \mathfrak{gl}_n^{(1)})$ is easy, using [5, Corollary (3.2)]. One finds that $H^2(\mathrm{GL}_{n,k}, \mathfrak{gl}_n^{(1)})$ is one-dimensional. Thus, any nonzero $c[1]$ is up to scaling equal to the Witt vector class.

6. Using the classes

We write G for $\mathrm{GL}_{n,k}$, the algebraic group GL_n over k. Sometimes it is instructive to restrict to SL_n or other reductive subgroups of GL_n. We leave this to the reader.

6.1. Other universal classes

We recall some constructions from [23]. If M is a finite-dimensional vector space over k and $r \geq 1$, we have a natural homomorphism between symmetric algebras $S^*(M^{\#(r)}) \to S^*(M^{\#(1)})$ induced by the map $M^{\#(r)} \to S^{p^{r-1}}(M^{\#(1)})$ which raises an element to the power p^{r-1}. It is a map of bialgebras. Dually, we have the bialgebra map $\pi^{r-1} : \Gamma^*(M^{(1)}) \to \Gamma^*(M^{(r)})$, whose kernel is the ideal generated by $\Gamma^1(M^{(1)})$ through $\Gamma^{p^{r-1}-1}(M^{(1)})$. So π^{r-1} maps $\Gamma^{p^{r-1}a}(M^{(1)})$ onto $\Gamma^a(M^{(r)})$.

6.1.1. Notation

We now introduce analogues of the classes e_r and $e_r^{(j)}$ of Friedlander and Suslin [9, Theorem 1.2, Remark 1.2.2]. We write $\pi_*^{r-1}(c[ap^{r-1}]) \in H^{2ap^{r-1}}(G, \Gamma^a(\mathfrak{gl}_n^{(r)}))$ as $c_r[a]$. Next we get $c_r[a]^{(j)} \in H^{2ap^{r-1}}(G, \Gamma^a(\mathfrak{gl}_n^{(r+j)}))$ by Frobenius twist. As in [9], a notation like $S^*(M(i))$ means the symmetric algebra $S^*(M)$, but graded, with M placed in degree i.

Here is the analogue of [23, Lemma 4.7].

LEMMA 6.2

The $c_i[a]^{(r-i)}$ enjoy the following properties ($r \geq i \geq 1$).

(1) *There is a homomorphism of graded algebras $S^*(\mathfrak{gl}_n^{\#(r)}(2p^{i-1})) \to H^{2p^{i-1}*}(G_r, k)$ given on $\mathfrak{gl}_n^{\#(r)}(2p^{i-1}) = H^0(G_r, \mathfrak{gl}_n^{\#(r)})$ by cup product with the restriction of $c_i[1]^{(r-i)}$ to G_r. If $i = 1$, then it is given on $S^a(\mathfrak{gl}_n^{\#(r)}(2)) = H^0(G_r, S^a(\mathfrak{gl}_n^{\#(r)}))$ by cup product with the restriction of $c[a]^{(r-1)}$ to G_r.*

(2) *For $r \geq 1$, the restriction of $c_r[1]$ to $H^{2p^{r-1}}(G_1, \mathfrak{gl}_n^{(r)})$ is nontrivial, so that $c_r[1]$ may serve as the universal class e_r in [9, Theorem 1.2].*

Proof

When M is a G-module, one has a commutative diagram

$$\begin{array}{ccc} \Gamma^m M \otimes \bigotimes^m M^\# & \to & \bigotimes^m M \otimes \bigotimes^m M^\# \\ \downarrow & & \downarrow \\ \Gamma^m M \otimes S^m M^\# & \to & k \end{array}$$

Take $M = \mathfrak{gl}_n^{(1)}$. There is a homomorphism of algebras $\bigotimes^*(\mathfrak{gl}_n^{\#(1)}) \to H^{2*}(G_1, k)$ given on $\mathfrak{gl}_n^{\#(1)}$ by cup product with $c[1]$. (We do not mention obvious restrictions to subgroups like G_1 anymore.) On $\bigotimes^m(\mathfrak{gl}_n^{\#(1)})$, it is given by cup product with $c[1] \cup \cdots \cup c[1]$, so by Theorem 5.1 it is also given by cup product with $c[m]$, using the pairing $\Gamma^m M \otimes \bigotimes^m M^\# \to k$. As this pairing factors through $\Gamma^m M \otimes S^m M^\#$, we get that the induced algebra map $S^*(\mathfrak{gl}_n^{\#(1)}) \to H^{2*}(G_1, k)$ is given by cup product with $c[m]$ on $S^m(\mathfrak{gl}_n^{\#(1)})$. If we compose with the algebra map $S^*(\mathfrak{gl}_n^{\#(i)}) \to S^{p^{i-1}*}(\mathfrak{gl}_n^{\#(1)})$, we get an algebra map $\psi : S^*(\mathfrak{gl}_n^{\#(i)}) \to H^{2p^{i-1}*}(G_1, k)$ given on $\mathfrak{gl}_n^{\#(i)}$ by cup product with $c[p^{i-1}]$, using the pairing $\mathfrak{gl}_n^{\#(i)} \otimes \Gamma^{p^{i-1}}(\mathfrak{gl}_n^{(1)}) \to k$. This pairing factors through $\mathrm{id} \otimes \pi^{i-1} : \mathfrak{gl}_n^{\#(i)} \otimes \Gamma^{p^{i-1}}(\mathfrak{gl}_n^{(1)}) \to \mathfrak{gl}_n^{\#(i)} \otimes \mathfrak{gl}_n^{(i)}$, so the homomorphism ψ is given on $\mathfrak{gl}_n^{\#(i)}$ by cup product with $\pi_*^{i-1} c[p^{i-1}] = c_i[1]$. We can lift it to an algebra map $S^*(\mathfrak{gl}_n^{\#(i)}) \to H^{2p^{i-1}*}(G_i, k)$ simply by still using the cup product with $c_i[1]$ on $\mathfrak{gl}_n^{\#(i)}$. Pull back along the $(r-i)$th Frobenius homomorphism $G_r \to G_i$ and you get an algebra map $\psi^{(r-i)} : S^*(\mathfrak{gl}_n^{\#(r)}) \to H^{2p^{i-1}*}(G_r, k)$, given on $\mathfrak{gl}_n^{\#(r)}$ by cup product with $c_i[1]^{(r-i)}$. If $i = 1$, pull back the cup product with $c[m]$ on $S^m(\mathfrak{gl}_n^{\#(1)})$ to a cup product with $c[m]^{(r-1)}$ on $S^m(\mathfrak{gl}_n^{\#(r)}(2))$. This then describes the homomorphism $\psi^{(r-1)}$ degree-wise.

In fact, if we restrict $c_r[1]$ as in [23, Remark 4.1] to $H^{2p^{r-1}}(\mathbb{G}_{a1}, (\mathfrak{gl}_n^{(r)})_{p^r\alpha}) = H^{2p^{r-1}}(\mathbb{G}_{a1}, k) \otimes (\mathfrak{gl}_n^{(r)})_{p^r\alpha}$, then even that restriction is nontrivial. That is because the Witt vector class generates the polynomial ring $H^{\mathrm{even}}(\mathbb{G}_{a1}, k)$ (see [12, Part I, Section 4.26]). And at this level $\Gamma^m \hookrightarrow \bigotimes^m$ gives an isomorphism, showing that $c[m]$ restricts to the mth power of the polynomial generator. □

6.2. Noetherian homomorphisms

Let A be a commutative k-algebra. The cohomology algebra $H^*(G, A)$ is then graded commutative, so we must also consider graded commutative algebras.

Definition 6.3

If $f : A \to B$ is a homomorphism of graded commutative k-algebras, we call f *Noetherian* if f makes B into a Noetherian left A module.

Remark 6.4

In algebraic geometry, a Noetherian homomorphism between finitely generated commutative k-algebras is called a *finite* morphism. With our terminology, we wish to stress the importance of chain conditions in our arguments.

LEMMA 6.5

The composite of Noetherian homomorphisms is Noetherian.

Proof

If $A \to B$ and $B \to C$ are Noetherian, view C as a quotient of the module B^r for some r. □

LEMMA 6.6

If the composite of $A \to B$ and $B \to C$ is Noetherian, so is $B \to C$.

Proof

View B-submodules of C as A-modules. □

Remark 6.7

In Lemma 6.6, $A \to C$ and $B \to C$ must be homomorphisms, but $A \to B$ could be just a map.

LEMMA 6.8

Suppose that B is finitely generated as a graded commutative k-algebra. Then $f : A \to B$ is Noetherian if and only if B^{even} is integral over $f(A^{\text{even}})$.

Proof

The map $B^{\text{even}} \to B$ is Noetherian. So if B^{even} is integral over $f(A^{\text{even}})$, then f is Noetherian. Conversely, if f is Noetherian and $b \in B^{\text{even}}$, then for some r one must have $b^r \in \sum_{i<r} f(A)b^i$. But then in fact $b^r \in \sum_{i<r} f(A^{\text{even}})b^i$. □

In particular, one has the following.

LEMMA 6.9
Suppose that B is a finitely generated commutative k-algebra. Let $n > 1$, and let A be a subalgebra of B containing x^n for every $x \in B$. Then $A \hookrightarrow B$ is Noetherian, and A is also finitely generated.

Proof
We follow Noether [17]. Indeed, B is integral over A. Take finitely many generators b_i of B, and let C be the subalgebra generated by the b_j^n. Then A is a C-submodule of B and hence finitely generated. □

Let us recall a result of invariant theory.

LEMMA 6.10 ([11, Theorem 16.9])
Let $f : A \to B$ be a Noetherian homomorphism of finitely generated graded commutative k-algebras with rational G-action. Then $A^G \to B^G$ is Noetherian. □

LEMMA 6.11
Let $f : A \to B$ be a Noetherian homomorphism of finitely generated graded commutative k-algebras with rational G_r-action. Then $H^*(G_r, A) \to H^*(G_r, B)$ is Noetherian.

Proof
Take $C = H^0(G_r, A^{\text{even}})$ or its subalgebra generated by the p^rth powers in A^{even}. Then apply [9, Theorem 1.5, Remark 1.5.1]. □

We need a minor variation on a theorem of Friedlander and Suslin.

THEOREM 6.12 ([9, Theorem 1.5])
Let $r \geq 1$. Let $S \subset G_r$ be an infinitesimal group scheme over k of height at most r. Further, let C be a finitely generated commutative k-algebra (considered as a trivial S-module), and let M be a Noetherian C-module on which S acts by C-linear transformations. Then $H^*(S, M)$ is a Noetherian module over the algebra $\bigotimes_{i=1}^r S^*((\mathfrak{gl}_n^{(r)})^\#(2p^{i-1})) \otimes C$, with the map given as suggested by Lemma 6.2.

COROLLARY 6.13
The restriction map $H^*(G_r, C) \to H^*(G_{r-1}, C)$ is Noetherian.

Proof
Take $S = G_{r-1}$, and note that the map $\bigotimes_{i=1}^{r} S^*((\mathfrak{gl}_n^{(r)})^\#(2p^{i-1}))\otimes C \to H^*(G_{r-1}, C)$ factors through $H^*(G_r, C)$. □

Proof of Theorem 6.12
The key difference with [9, Theorem 1.5, Remark 1.5.1] is that we do not require the height of S to be r. (As $S \subset G_r$, the fact that its height is at most r is automatic.) Thus, to start their inductive argument, we must also check the obvious case where $r = 1$ and S is the trivial group. The rest of the proof goes through without change. □

Remark 6.14
If S has height s, then the map $(\mathfrak{gl}_n^{(r)})^\#(2p^{i-1}) \to H^{2p^{i-1}}(S, k)$ is trivial for $r - i \geq s$.

6.3. Cup products on the cochain level
As we need a differential graded algebra structure on Hochschild-Serre spectral sequences, we now expand the discussion of the Hochschild complex in [12, Part I, Section 5.14]. Let L be an affine algebraic group scheme over the field k, let N be a normal subgroup scheme, and let R be a commutative k-algebra on which L acts rationally by algebra automorphisms. We have a Hochschild complex $C^*(L, R)$ with $R\otimes k[L]^{\otimes i}$ in degree i. Define a cup product on $C^*(L, R)$ as follows. If $u \in C^r(L, R)$ and $v \in C^s(L, R)$, then $u \cup v$ is defined in simplified notation by

$$(u \cup v)(g_1, \dots, g_{r+s}) = u(g_1, \dots, g_r)^{g_1\cdots g_r} v(g_{r+1}, \dots, g_{r+s}),$$

where ${}^g r$ denotes the image of $r \in R$ under the action of g. The following lemma is easy to check.

LEMMA 6.15
With this cup product, $C^*(L, R)$ is a differential graded algebra.

In particular, taking for R the algebra $k[L]$ with L acting by right translation, we get the differential graded algebra $C^*(L, k[L])$, quasi-isomorphic to k. And the action by left translation on $k[L]$ is by L-module isomorphisms, so this makes $C^*(L, k[L])$ into a differential graded algebra with L-action. It consists of injective L-modules in every degree. We write $C^*(L)$ for this differential graded algebra with L-action. One has $C^i(L) = k[L]^{\otimes i+1}$, and this is our elaboration of [12, Part I, Section 4.15(1)].

The Hochschild-Serre spectral sequence

$$E_2^{rs} = H^r\big(L/N, H^s(N, R)\big) \Rightarrow H^{r+s}(L, R)$$

can now be based on the double complex $\big(C^*(L/N)\otimes(C^*(L)\otimes R)^N\big)^{L/N}$. The tensor product over k of two differential graded algebras is again a differential graded algebra, and the spectral sequence inherits differential graded algebra structures [2, Section 3.9] from such structures on $C^*(L)\otimes R$, $(C^*(L)\otimes R)^N$, $C^*(L/N)\otimes(C^*(L)\otimes R)^N$.

6.4. Hitting invariant classes

We now come to the main result of this section, which is the counterpart of [23, Corollary 4.8]. It does not seem to follow from the CFG conjecture, but we show that it implies the conjecture.

THEOREM 6.16

Let $r\geq 1$. Further, let A be a finitely generated commutative k-algebra with G-action. Then $H^{\mathrm{even}}(G,A)\to H^0(G,H^*(G_r,A))$ is Noetherian.

Remark 6.17

Recall that $H^0(G,H^*(G_r,A))$ is finitely generated as a k-algebra, by [9] and invariant theory.

Proof of Theorem 6.16

Step 1. If M is a G-module on which G_r acts trivially, then $H^0(G,M)$ and $H^0(G/G_r,M)$ denote the same subspace of M. We may thus switch between these variants.

Step 2. We argue by induction on r. Put $C=H^0(G_r,A)$. Then C contains the elements of A raised to the power p^r, so C is also a finitely generated algebra and A is a Noetherian module over it.

Step 3. Let $r=1$. This case is the same as in [23]. By [9, Theorem 1.5], $H^*(G_1,A)$ is a Noetherian module over the finitely generated algebra

$$R=S^*\big((\mathfrak{gl}_n^{(1)})^\#(2)\big)\otimes C.$$

Then, by invariant theory (see [11, Theorem 16.9]), $H^0(G,H^*(G_1,A))$ is a Noetherian module over the finitely generated algebra $H^0(G,R)$. By Lemma 6.2, we may take the algebra homomorphism $R\to H^*(G_1,A)$ of [9] to be based on cup product with our $c[a]=c[a]^{(0)}$ on the summand $S^a((\mathfrak{gl}_n^{(1)})^\#(2))\otimes C$. But then the map $H^0(G,R)\to H^*(G_1,A)$ factors, as a linear map, through $H^{\mathrm{even}}(G,A)$. This settles the case $r=1$ by Remark 6.7.

Step 4. Now let the level r be greater than one. We follow the analysis in [9, Section 1] to peel off one level at a time. Heuristically, in the tensor product of Theorem 6.12, we treat one factor at a time. That is the main difference with the argument in [23].

Thus, consider the Hochschild-Serre spectral sequence $E_2^{ij}(C) = H^i(G_r/G_{r-1}, H^j(G_{r-1}, C)) \Rightarrow H^{i+j}(G_r, C)$. We first wish to argue that this spectral sequence stops, meaning that $E_s^{**}(C) = E_\infty^{**}(C)$ for some finite s. This is proved in [9, Section 1] for a very similar spectral sequence. So we imitate the argument. We need to apply [9, Lemma 1.6] and its proof; we use $H^{\text{even}}(G_r, C)$ for the A of that lemma and $S^*((\mathfrak{gl}_n^{(r)})^\#(2))$ for its B. We map $H^{\text{even}}(G_r, C)$ in the obvious way to the abutment $H^*(G_r, C)$, and for $B \to E_2^{*0}(k) = H^*(G_r/G_{r-1}, k) = H^*(G_1, k)^{(r-1)}$ we use the $(r-1)$st Frobenius twist of the map $S^*((\mathfrak{gl}_n^{(1)})^\#(2)) \to H^*(G_1, k)$ of Lemma 6.2. So we use the class $c[a]^{(r-1)}$ on $S^a((\mathfrak{gl}_n^{(r)})^\#(2))$. By Corollary 6.13 and Lemma 6.8, the restriction map $H^{\text{even}}(G_r, C) \to H^*(G_{r-1}, C)$ is Noetherian, and by Theorem 6.12 (cf. [9, Corollary 1.8]), it follows that the $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))$ module $E_2^{**}(C) = H^*(G_1, H^*(G_{r-1}, C)^{(1-r)})^{(r-1)}$ is Noetherian, so the spectral sequence stops, say, at $E_s^{**}(C)$. Note also that the image of $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))$ in $E_2^{**}(C)$ consists of permanent cycles.

Step 5. As the spectral sequence is one of graded commutative differential graded algebras, the pth power of an even cochain in a page passes to the next page. As the spectral sequence stops at page s, one finds that, for an $x \in E_2^{\text{even,even}}(C)$, the power x^{p^s} is a permanent cycle. Let P be the algebra generated by permanent cycles in $E_2^{\text{even,even}}(C)$. Then $P \to E_t^{ij}(C)$ is Noetherian for $2 \le t \le \infty$. So $P^G \to (E_\infty^{**}(C))^G$ is Noetherian by Lemma 6.10.

Step 6. By the inductive assumption, $H^0\big(G, H^*(G_{r-1}, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))\big)$ is Noetherian over $H^{\text{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$. By step 1, we may rewrite $H^0\big(G, H^j(G_{r-1}, C \otimes S^i((\mathfrak{gl}_n^{(r)})^\#(2)))\big)$ as $H^0\big(G/G_{r-1}, H^j(G_{r-1}, C \otimes S^i((\mathfrak{gl}_n^{(r)})^\#(2)))\big)$. The latter description will be needed in the sequel. We may map $H^0\big(G/G_{r-1}, H^j(G_{r-1}, C \otimes S^i((\mathfrak{gl}_n^{(r)})^\#(2)))\big)$ by restriction to $H^0\big(G_r/G_{r-1}, H^j(G_{r-1}, C) \otimes S^i((\mathfrak{gl}_n^{(r)})^\#(2))\big)$ and then to $E_2^{2i,j}(C) = H^{2i}(G_r/G_{r-1}, H^j(G_{r-1}, C))$ by cup product with $c[i]^{(r-1)}$. So we now have a map from $H^{\text{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$ to $E_2^{**}(C)$. We factor it further.

Step 7. One checks that the map from $H^{\text{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$ to $H^0\big(G_r/G_{r-1}, H^*(G_{r-1}, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$ of step 6 factors naturally through the algebra $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))$ of step 4. Moreover, as the algebra $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))$ acts on the full spectral sequence, we may make $H^{\text{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$ act on the full spectral sequence by way of that algebra.

Step 8. The Noetherian map $H^{\text{even}}(G_r, C) \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)) \to E_2^{**}(C)$ factors through $H^0\big(G_r/G_{r-1}, H^*(G_{r-1}, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))\big)$. But then $H^0\big(G_r/G_{r-1}, H^*(G_{r-1}, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))\big) \to E_2^{**}(C)$ is Noetherian by Lemma 6.6. So by Lemma 6.10, the map $H^0\big(G/G_{r-1}, H^*(G_{r-1}, C\otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2)))\big) \to (E_2^{**}(C))^G$ is Noetherian. Combining with the inductive hypothesis, we learn that $H^{\mathrm{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big) \to (E_2^{**}(C))^G$ is Noetherian. It lands in P^G because the map in step 4 lands in P. We conclude that $H^{\mathrm{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big) \to (E_\infty^{**}(C))^G$ is Noetherian.

Step 9. We filter $H^{\mathrm{even}}\big(G, C\otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$ by putting $H^{\mathrm{even}}\big(G, C\otimes S^t((\mathfrak{gl}_n^{(r)})^\#(2))\big)$ in $H^{\mathrm{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)^{\geq j}$ for $t \geq j$. As in [9, Section 1], the filtered algebra may be identified with its associated graded, and the map $H^{\mathrm{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big) \to H^*(G_r, C)$ respects filtrations. Now we care about $H^*(G_r, C)^G$ as a module for $H^{\mathrm{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$. To see that it is Noetherian, we may pass as in [9, Section 1] to the associated graded, where one puts $(H^*(G_r, C)^G)^{\geq j} = (H^*(G_r, C)^G) \cap H^*(G_r, C)^{\geq j}$. This associated graded of $H^*(G_r, C)^G$ is a submodule of $(E_\infty^{**}(C))^G$, containing the image of $H^{\mathrm{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$. So it is indeed Noetherian. We conclude that $H^*(G_r, C)^G$ is Noetherian over $H^{\mathrm{even}}\big(G, C \otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big)$.

Step 10. As in the case $r = 1$, the map $H^{\mathrm{even}}\big(G, C\otimes S^*((\mathfrak{gl}_n^{(r)})^\#(2))\big) \to H^*(G_r, C)^G$ factors, as a linear map, through $H^{\mathrm{even}}(G, C)$, so $H^{\mathrm{even}}(G, C) \to H^*(G_r, C)^G$ is Noetherian by Remark 6.7. As $H^*(G_r, C)^G \to H^*(G_r, A)^G$ is Noetherian, the result follows. □

6.5. Cohomological finite generation (CFG)

Now let A be a finitely generated commutative k-algebra with G-action. We wish to show that $H^*(G, A)$ is finitely generated, following the same path as in [23] but using improvements from [20]. As in [23], we denote by $\mathscr{A}$ the coordinate ring of a flat family with general fiber A and special fiber $\mathrm{gr}\, A$ (see [10, Theorem 13]). Choosing r as in [23, Proposition 3.8], we have the spectral sequence

$$E_2^{ij} = H^i\big(G/G_r, H^j(G_r, \mathrm{gr}\, A)\big) \Rightarrow H^{i+j}(G, \mathrm{gr}\, A),$$

and $R = H^*(G_r, \mathrm{gr}\, A)^{(-r)}$ is a finite module over the algebra

$$\bigotimes_{a=1}^{r} S^*\big((\mathfrak{gl}_n)^\#(2p^{a-1})\big) \otimes \mathrm{hull}_\nabla(\mathrm{gr}\, A).$$

This algebra has a good filtration, and by the main result of [20], the ring R has finite good filtration dimension. In particular, there are only finitely many nonzero $H^i(G, R)$. Thus, the same main result says that $E_2^{0*} \to E_2^{**}$ is Noetherian. Now $H^0(G/G_r, H^*(G_r, \mathscr{A})) \to H^0(G/G_r, H^*(G_r, \mathrm{gr}\, A))$ is Noetherian by [9] and Lemma 6.10. And $H^*(G, \mathscr{A}) \to H^0(G/G_r, H^*(G_r, \mathscr{A}))$ is Noetherian by

Theorem 6.16, so another application of [9, Lemma 1.6] (with $B = k$) shows that $H^*(G, \mathscr{A}) \to H^*(G, \operatorname{gr} A)$ is Noetherian.

There is a map of spectral sequences from a totally degenerate spectral sequence

$$E(\mathscr{A}) : \quad E_1^{ij}(\mathscr{A}) = H^{i+j}(G, \operatorname{gr}_{-i} \mathscr{A}) \Rightarrow H^{i+j}(G, \mathscr{A}),$$

with pages $H^*(G, \mathscr{A})$, to the spectral sequence

$$E(A) : \quad E_1^{ij}(A) = H^{i+j}(G, \operatorname{gr}_{-i} A) \Rightarrow H^{i+j}(G, A).$$

This makes $H^*(G, \mathscr{A})$ act on $E(A)$, and the Noetherian homomorphism $H^*(G, \mathscr{A}) \to H^*(G, \operatorname{gr} A)$ is used in standard fashion (see [24, Slogan 3.9]) to make the spectral sequence $E(A)$ stop. It follows easily that $H^*(G, A)$ is finitely generated. So far G was $\mathrm{GL}_{n,k}$. As explained in some detail in [24], this case implies our CFG conjecture (over fields.)

Remark 6.18

The spectral sequence $E(A)$ is based on filtering the Hochschild complex of A. As it lives in the second quadrant, the exposition of multiplicative structure in [2, Section 3.9] does not apply as stated. (In order to avoid convergence issues, [2] uses a filtration that reaches a maximum.) But [4, Chapter XV, Example 2] is sufficiently general to cover our case (or see [14]).

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