PRESENTING K_2 WITH GENERIC SYMBOLS

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ABSTRACT.

Given an ideal in a ring with many units, we give a presentation for the relative K_2 in terms of symbols whose entries are in general position.

0. INTRODUCTION

Let R be a commutative ring with many units. (Say R is unit-irreducible in the sense of [1]). Let I be an ideal of R. Then $K_2(R)$ has a presentation with Steinberg symbols and $K_2(R,I)$ may be described in a similar manner, by Keune/Loday. ([2],[3]). Marc Levine asked to give a presentation in terms of symbols whose entries are in general position. That this would be possible is suggested by some proofs in [1]. Here we work out such a presentation with a suitably formalized notion of general position.

1. CONVENTIONS

Let R be a commutative ring, I an ideal in R. (It could be the unit ideal). Put $R^* = GL_1(R)$. We are given a subset G of

J.F. Jardine and V.P. Snaith (eds.), Algebraic K-Theory: Connections with Geometry and Topology, 509–516.

$\{(r,x) | r \in R^*, x \in I, 1 - rx \in R^*\}$.

Here G stands for "generic". We now formalize that G should have many elements. (This is analogous to the existence of many units in [1]). First of all we require that G is not empty. Next we impose the following axiom (analogous to unit-irreducibility in [1]).

Suppose $n \geq 1$ is an integer and for $1 \leq i \leq n$ one is given $f_i, h_i \in R[X,Y]$, $g_i \in IR[X,Y]$, $r_i \in R$, $x_i \in I$ such that $h_i(r_i,x_i) \in R^*$ and $(f_i(r_i,x_i)/h_i(r_i,x_i), g_i(r_i,x_i)/h_i(r_i,x_i)) \in G$.

Then the axiom requires the existence of $r \in R$, $x \in I$ such that for all i between 1 and n simultaneously one has $h_i(r,x) \in R^*$ and $(f_i(r,x)/h_i(r,x), g_i(r,x)/h(r,x)) \in G$.

The axiom requires the existence of r,x whenever one is given such data. In order to get some practice with the use of this kind of axiom one may read [1].

2. EXAMPLE

Let R be a local ring with infinite residue field and maximal ideal $\underline{m}$. Let I be a non-zero principal proper ideal. Put $G = \{(r,x) | r \in R^*, x \in I, x \notin \underline{m} I\}$. One easily checks that the axiom is satisfied.

3. THEOREM

Let R,I,G satisfy the conditions in the conventions. Then $K_2(R,I)$ has the following presentation.

Generators are the $\langle r,x\rangle$ with $(r,x) \in G \cup (\{1\} \times I) \cup (R^* \times \{0\})$.

Relations are

(R0) The group is abelian

(R1) $\langle r_1,x\rangle + \langle r_2,x\rangle = \langle r_1+r_2-r_1r_2x,x\rangle$

(R2) $\langle r,x_1\rangle + \langle r,x_2\rangle = \langle r,x_1+x_2-rx_1x_2\rangle$

(R3) $\langle r_1,r_2x\rangle + \langle r_2,r_1x\rangle = \langle r_1r_2,x\rangle$.

Here r,r_1,r_2 are in R and x,x_1,x_2 in I. Moreover a relation applies if and only if all its terms are defined.

4. REMARK

Observe that there is no relation of the type $\langle a,b\rangle + \langle b,a\rangle = 0$ in the list. One reason is that in example 2 one sees that it may never occur that both (a,b) and (b,a) are in G. Similarly the Steinberg relation is hidden, as it now involves more than one generator. (This in case $R = I$).

5. Observe that $\langle 1,x\rangle = 0$ for $x \in I$ because of (R3). Similarly $\langle r,0\rangle = 0$ for $r \in R^*$, by (R2). Nevertheless these dummy generators serve a purpose. For instance, suppose (r,x_1), (r,x_2) are in G and $x_1 + x_2 = rx_1x_2$. Then (R2) tells $\langle r,x_1\rangle + \langle r,x_2\rangle = \langle r,0\rangle = 0$.

On the other hand, the $\langle 1,x\rangle$ for which $1 - x$ is not a unit are artificial and may be deleted.

6. We start the proof of the theorem.

We know by Keune [2] how to get a presentation for $K_2(R,I)$ from one for $K_2(R \ltimes I)$, where $R \ltimes I = \{(r,s) \mid r \in R,\ r-s \in I) \subseteq R \times R$ is the "double". Now it is easy to see from the axiom for G that $R \ltimes I$ has many units, so that its K_2 has the ordinary presentation ("theorem of Matsumoto") with Steinberg symbols. ([1]) Theorem 3.4 or Corollary 8.5). Thus we get the following presentation for $K_2(R,I)$.

Generators: The symbols $\{(r,s),\ (u,v)\}$ with $(r,s),\ (u,v) \in (R \ltimes I)^*$.

Relations:

(i) The group is abelian

(ii) $\{a,bc\} = \{a,b\} + \{a,c\}$

(iii) $\{ab,c\} = \{a,c\} + \{b,c\}$

(iv) $\{a,1-a\} = 0$

(v) $\{(r,r),\ (s,s)\} = 0$

(vi) $\{(r,1),\ (1,s)\} = 0.$

Here, as usual, the relations apply only when the terms are defined. Thus in (vi) one must have $r-1,\ s-1 \in I$, and in (ii) one must have $a,b,c \in (R \ltimes I)^*$. Now if $D_{gen}(R,I)$ denotes the group given by the presentation in the theorem, then we have of course a homomorphism $D_{gen}(R,I) \longrightarrow K_2(R,I)$, sending $\langle r,x \rangle$ to $\{(r,r),\ (1,\ 1-rx)\}$ for $(r,x) \in G$. We seek an inverse homomorphism.

7. For the time being we compute in $D_{gen}(R,I)$.

Put $\{(r,s),\ (u,v)\} = \langle s, s^{-1}-s^{-1}u^{-1}v \rangle - \langle u, u^{-1}-u^{-1}r^{-1}s \rangle$ whenever

(r,s), $(u,v) \in (R \ltimes I)^*$ are such that $(s, s^{-1}-s^{-1}u^{-1}v)$ and $(u, u^{-1}-u^{-1}r^{-1}s)$ are in G. The idea is that this is the case generically. That is, this gives conditions on $r \in R$, $s-r \in I$, $u \in R$, $v-u \in I$ that can be satisfied simultaneously with other conditions of a similar nature. (Check this). This is what we will mean when we use the term "generic". Compare [1]. We claim that relations (ii), (iii), (iv) hold generically. Take relation (iv) for instance. We have

$$\{(r,s),(1-r,1-s)\} = \langle s, s^{-1}-s^{-1}(1-r)^{-1}(1-s)\rangle - \langle 1-r,(1-r)^{-1}(1-r^{-1}s)\rangle$$

and our first worry is if this makes sense generically. Now $\langle s,x\rangle$ and $\langle 1-s,x\rangle$ make sense generically. Thus we can (generically) choose $r = 1-(1-sx)^{-1}(1-s)$ to get $(s, s^{-1}-s^{-1}(1-r)^{-1}(1-s)) = (s,x) \in G$, so that $\langle s, s^{-1}-s^{-1}(1-r)^{-1}(1-s)\rangle$ is OK at least once, and therefore generically. The other term is treated similarly. We want to show the two terms in the definition of $\{(r,s), (1-r,1-s)\}$ cancel generically. Indeed one has generically

$$\langle s, s^{-1}-s^{-1}(1-r)^{-1}(1-s)\rangle + \langle 1-r, s^{-1}-s^{-1}(1-r)^{-1}(1-s)\rangle =$$
$$\langle 1, s^{-1}-s^{-1}(1-r)^{-1}(1-s)\rangle = 0 \quad \text{and}$$
$$\langle 1-r,(1-r)^{-1}(1-r^{-1}s)\rangle + \langle 1-r, s^{-1}-s^{-1}(1-r)^{-1}(1-s)\rangle = \langle 1-r,0\rangle = 0.$$

(The auxiliary term $\langle 1-r, s^{-1}-s^{-1}(1-r)^{-1}(1-s)\rangle$ exists generically).

8. Let $a,b \in (R \ltimes I)^*$. We put $\{a,b\}(p,q) = \{ap,bq\} - \{p,bq\} - \{ap,q\} + \{p,q\}$ with p,q chosen generically so that the right hand side makes sense in $D_{gen}(R,I)$. Now

$\{a,b\}(p,q)$ is independent of the generically chosen p,q, because the generic validity of relations (ii), (iii) easily implies that $\{a,b\}(p,q) = \{a,b\}(pp_1,qq_1)$ for generic p_1,q_1. Therefore we often write $\{a,b\}$ for $\{a,b\}(p,q)$. (Check that there is no conflict with earlier notation). We claim that relations (ii), (iii) hold, not just generically. This follows for similar reasons. For instance, if $a,b,c \in (R \ltimes I)^*$, then one gets

$$\{a_1a_2,b\} - \{a_1,b\} - \{a_2,b\} = \{a_1p_1a_2p_2,bq\} - \{p_1p_2,bq\}$$
$$- \{a_1p_1a_2p_2,q\} + \{p_1p_2,q\} - \{a_1p_1,bq\} + \{p_1,bq\} + \{a_1p_1,q\}$$
$$- \{p_1,q\} - \{a_2p_2,bq\} + \{p_2,bq\} + \{a_2p_2,q\} - \{p_2,q\} = 0,$$

where p_1,p_2,q are chosen generically. Thus the symbol $\{a,b\}$ is now bilinear, but we only know $\{a,1-a\}$ for generic a. For generic a we also have $0 = \{a^{-1},1-a^{-1}\} = \{a,(a-1)^{-1}a\} = \{a,-a\}$. Thus for a and b generic we have $\{a,b\} + \{b,a\} = \{ab,-ab\} - \{a,-a\} - \{b,-b\} = 0$. But then $\{a,b\} + \{b,a\}$ must vanish for all $a,b \in (R \ltimes I)^*$ because of bilinearity of $(a,b) \mapsto \{a,b\} + \{b,a\}$. Then it is also clear that $\{a,-a\}$ vanishes for all $a \in (R \ltimes I)^*$. Now suppose $a \in R \ltimes I$ is such that $a(1-a) \in (R \ltimes I)^*$. For generic x the symbols $\{(1-ax)^{-1}a(1-x),\ (1-ax)^{-1}(1-a)\}$, $\{ax,1-ax\}$, $\{(x-1)^{-1}x(1-a),\ (1-x)^{-1}(1-ax)\}$, $\{x,1-x\}$ all vanish because of (iv). Add the first two and then subtract the last two. With what we have learned this easily yields $\{a,1-a\} = 0$, which thus actually holds for all relevant a. Therefore we get a homomorphism $K_2(R \ltimes I) \longrightarrow D(R,I)$ sending $\{a,b\}$ to $\{a,b\}$.

9. To get an inverse for the map $D_{gen}(R,I) \longrightarrow K_2(R,I)$ we simply send $\{(r,s),(u,v)\}$ to $\{(r,s),(u,v)\}$. We know already that this respects relations (i), (ii), (iii), (iv). One checks it respects (v), (vi) too. It is not difficult to finish the proof of the theorem.

10. EXERCISE

Let R be a field. Take $I = R$ and $G = \{(r,s) \in R^* \times R \mid 1-rs \in R^*\}$. Show directly from Matsumoto's theorem that the presentation in the theorem is valid. If the artificial generator $\langle 1,1 \rangle$ is removed, show that the presentation still works if R is not the field with three elements.

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