A NOTE ON EXCISION FOR K_2

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Summary. We consider Ruth Charney's excision theorem ([C]) for the special case of K_2. We describe a different proof for this special case. It follows from this proof that if one requires excision only on the K_2 level, one may weaken the condition somewhat. (Recall from [S] that the same holds if one requires excision only on the K_1 level). We also give a counterexample to a stronger statement: We give an example of an ideal J with $J = J^2$ for which excision fails.

1.1. Let J be an associative ring without unit and let m be a positive integer. Following Charney we say that J is an m-excision ideal for K_2 if the following holds: For any associative ring with unit A that contains J as a (2-sided) ideal, the map

$$K_2(\mathbb{Z}\oplus J,J)\otimes\mathbb{Z}\,[\tfrac{1}{m}] \to K_2(A,J)\otimes\mathbb{Z}\,[\tfrac{1}{m}]$$

is an isomorphism.(1)

1.2. Recall that $J\otimes_J J$ is the quotient of $J\otimes_{\mathbb{Z}} J$ by the subgroup generated by the elements $xy\otimes z - x\otimes yz$ with $x,y\ z\in J$. Multiplication in J defines a map

$$\mu : J\otimes_J J\otimes\mathbb{Z}\,[\tfrac{1}{m}] \to J\otimes\mathbb{Z}\,[\tfrac{1}{m}]$$

Theorem. (cf.[C]). If μ is a bijection then J is an m-excision ideal for K_2.

1.3. Proof. Assume the hypothesis of the theorem. Let J be an ideal in A. We view $K_2(A,J)$ as the kernel of $St(A,J) \to GL(J)$ where $St(A,J)$ is the group $St(A^{(\infty)},J,A^{(\infty)})$ of the appendix to [V]. It follows from [G - W] that

$$K_2(\mathbb{Z}\oplus J,J)\otimes\mathbb{Z}\,[\tfrac{1}{m}] \to K_2(A,J)\otimes\mathbb{Z}\,[\tfrac{1}{m}]$$

is surjective. Remains to show that the kernel of $K_2(\mathbb{Z}\oplus J,J) \to K_2(A,J)$ is m-torsion. Fix an element α of this kernel. Put

$$J_s = \{j \in J \mid m^s j \in J^2\}$$

so that J is the union of the increasing sequence of ideals J_s. Choose s so that α comes from $K_2(\mathbb{Z}\oplus J_s,J_s)$ and has trivial image in $St(A,J_s)$. Define $\overline{St}(\mathbb{Z}\oplus J)$ to be the quotient of $St(\mathbb{Z}\oplus J)$ by the m-torsion subgroup of $K_2(\mathbb{Z}\oplus J)$. The theorem follows from:

1.4. Lemma. There is a set theoretical map $\tau : St(A,J_s) \to \overline{St}(\mathbb{Z}\oplus J)$ such that the composite with $K_2(\mathbb{Z}\oplus J_s,J_s) \to St(A,J_s)$ equals m^s times the natural map $K_2(\mathbb{Z}\oplus J_s,J_s) \to \overline{St}(\mathbb{Z}\oplus J)$.

Proof of Lemma. Let F be the free group on G(F), where G(F) is the generating set used in the definition of $St(A^{(\infty)},J_s,A^{(\infty)})$, and let $R = \ker(F \to St(A,J_s))$. For each $x \in G(F)$, the m^s-th power of its matrix image lies in the image $E(\mathbb{Z}\oplus J)$ of $St(\mathbb{Z}\oplus J)$. Replacing m by m^s we may and shall further assume $s = 1$. Given n distinct elements $x_1,\dots,x_n$ in G(F), choose an integer N such that the matrix images $mat(x_i)$ all lie in $GL_N(J) \subseteq GL(J)$. Choose $y_i \in \overline{St}(\mathbb{Z}\oplus J)$ with matrix image

$$\begin{pmatrix} mat(x_i) & 0 & 0 \\ 0 & 1_{iN} & 0 \\ 0 & 0 & mat(x_i)^{-1} \end{pmatrix}$$

and define a homomorphism $<x_1,\dots,x_n> \to \overline{St}(\mathbb{Z}\oplus J)$ sending x_i to y_i. Restrict this homomorphism to the commutator subgroup of $<x_1,\dots,x_n>$. This restriction φ is characterised by the property:

Let $x,x' \in <x_1,\dots,x_n>$, $y,y' \in \overline{St}(\mathbb{Z}\oplus J), M \in \mathbb{N}$, such that the matrix image of y is

$$\begin{matrix} N \updownarrow \\ M \updownarrow \end{matrix} \begin{pmatrix} \mathrm{mat}(x) & 0 \\ 0 & P \end{pmatrix}$$

for some $P \in GL_M(J)$ and the matrix image of y' is

$$\begin{matrix} N \updownarrow \\ M \updownarrow \\ \end{matrix} \begin{pmatrix} \mathrm{mat}(x') & 0 & 0 \\ 0 & 1_M & 0 \\ 0 & 0 & Q \end{pmatrix}$$

for some $Q \in GL(J)$. Then $\varphi([x,x']) = [y,y']$.
(Compare the construction of Milnor's pairing in [M] §8 and use that we have factored out m-torsion in $K_2(\mathbb{Z}\oplus J)$, including the Steinberg symbols $\{\mathrm{mat}(x_i),\mathrm{mat}(x_i)\}$.)
Using this characterisation we extend φ to all of $[F,F]$ by varying $\{x_1,\dots,x_n\}$. Let H be the free subgroup of F generated by m-th powers of elements of G(F). For each $X(v,j,w)$ in G(F), choose p_i, $q_i \in J$ so that $\sum_i p_iq_i = mj$ (recall $s = 1$) and put

$$\psi(X(v,j,w)^m) = \prod_i X(vp_i,1,q_iw) \in \overline{St}(\mathbb{Z}\oplus J).$$

This defines a homomorphism $\psi : H \to \overline{St}(\mathbb{Z}\oplus J)$. It agrees with φ on $H \cap [F,F] = [H,H]$. We extend φ to $H[F,F]$ by putting $\varphi(xy) = \psi(x)\varphi(y)$ for $x \in H$, $y \in [F,F]$. Define $\tau : F \to \overline{St}(\mathbb{Z}\oplus J)$ by $\tau(x) = \varphi(x^m)$.
One shows that $\tau(x\,r\,y) = \tau(xy)\ \tau(r)$ for $x,y \in F$, $r \in R$.
Thus if τ annihilates R, τ factors through $St(A,J_s)$ and the lemma easily follows. To show that τ annihilates R indeed, one treats each of the defining relations listed in the appendix to [V]. To

deal with the third, for instance, recall the hypothesis of the theorem and use that for $X(v,j,w) \in G(F)$ there is a homomorphism

$$J \otimes_J J \to St(\mathbb{Z} \oplus J)$$

sending $p \otimes q$ to $X(vp,1,qw)$.

2. The counterexample. It is commutative. Put $R_r = \mathbb{Z}[T_r,\varepsilon]/(T^{2^r},\varepsilon^2)$. Embed R_r into R_{r+1} by sending T_r to T_{r+1}^2, ε to ε. Let $R = \varinjlim_r R_r$, $J = \varinjlim_r J_r$, with $J_r = T_r R_r$. Clearly $J = J^2$ so that

$J \otimes_J J \to J$ is surjective. Nevertheless $K_2(\mathbb{Z} \oplus J, J) \to K_2(R,J)$ is not injective: Consider $\alpha = \langle T_1, \varepsilon T_1 \rangle \langle \varepsilon T_1, -T_1 \rangle \in K_2(\mathbb{Z} \oplus J, J)$. Its image in $K_2(R,J)$ vanishes, by an easy computation. But suppose α vanishes. Then $\langle T_1, T_1\varepsilon \rangle \langle \varepsilon T_1, -T_1 \rangle$ must vanish in $K_2(\mathbb{Z} \oplus J_r)$ for some r. However, recall that we have a Chern class $K_2(\mathbb{Z} \oplus J_r) \to \Omega^2_{\mathbb{Z} \oplus J_r}$ sending $\langle a,b \rangle$ to $\pm(1 \pm ab)^{-1} da \wedge db$. (The reader may choose conventions and then determine the correct signs). Straightforward computation shows that the image in $\Omega^2_{\mathbb{Z} \oplus J_r}$ of our element is non-zero (This image is not even torsion).

(1) Charney has now replaced $\mathbb{Z}[\frac{1}{m}]$ by an arbitrary subring of $\mathbb{Q}$. Our theorem generalizes similarly.

References

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